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Oligopoly Competition with Reputation Manipulation

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Abstract

How does hidden review manipulation alter oligopoly market outcomes? We model strategic manipulation of reputations in price and quantity competitions. Each firm has private information about its product quality, which consumers need to infer before purchasing. Consumers observe a noisy review rating for each firm, which is the combination of authentic and manipulated (fake) reviews, and is thus the subject of strategic manipulation. A firm's review manipulation action, though costly, could inflate the rating of its product, increase consumers' willingness to pay, and upwardly shift the firm's demand function. By focusing on a linear strategy with private information, we establish a monotone equilibrium review manipulation strategy. When consumers rationally conjecture review manipulation, we show that expected prices and quantities become invariant to a non-manipulation environment, and severe competition diminishes the amount of manipulated reviews. Furthermore, due to the signaling role of manipulated reviews, expected consumer surplus increases. By contrast, when consumers naively believe observed ratings are genuine, strategic substitutability emerges among firms' review manipulation actions, and equilibrium oligopoly market outcomes are distorted accordingly. With naive consumers, severe competition can increase the amount of manipulated reviews and damage consumer surplus.

Key words: Reputation Manipulation, Fake Review, Oligopoly Market, Rating, Signal Jamming

JEL codes: D83, L13, L15, L51, L86

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1 Introduction

Despite considerable public concerns about hidden review manipulation, such as fake reviews, there are few theoretical studies modeling review manipulation with an oligopolistic market structure. Specifically, the number of economic articles that model strategic review manipulation is scarce, and this may hinder the regulatory discussions for anti-competitive policies.[1,2]

By proposing a review manipulation model under the classical Cournot and Bertrand differentiated product competition, this article aims to contribute to the literature and assist in policy discussions. Notably, for each firm, we introduce the asymmetric information of producer type (i.e., product quality for each firm).

Specifically, we contribute to the literature by reporting the review manipulating oligopoly equilibrium properties, welfare characterizations, and welfare comparisons, with a rational and naive consumer. We show that oligopoly market outcomes with rational consumers are drastically different from those with naive consumers, as the former involves consumer surplus improvement, while the latter is associated with damaged consumer surplus. In contrast to classic advertisement or investment competition in the literature, with a rational representative consumer, the firm's profit-maximizing review manipulation strategy does not depend on other firms' review manipulation strategy. This key property stems from the rational conjecturing process: A rational representative consumer, who observes a rating for each product, rationally conjectures the product quality of each project, and makes consumption choices. In equilibrium, given the review-manipulation schedules of each firm, a rational representative consumer downwardly assesses (or curve) the observed ratings and forms the expected product qualities. Based on this rational conjecturing process, at the simultaneous review-manipulating action stage (i.e., fake-review-writing stage), a firm chooses its review-manipulating action, anticipating that such consumer nullifies the *other* firms' manipulative behavior in expectation. In other words, the firm chooses its own review-manipulating strategy as if other firms are not making any review manipulation. Thus, there is essentially

[1] The pioneering and empirical study is Mayzlin et al. (2014), which reports a stylized Hotelling competition promotional (fake) review model in its Appendix. In their appendix, the authors assume that no economic agents (i.e., neither of the two firms at both ends of the Hotelling space, nor the consumers) know the true product quality until the end of the game (thus, information is incomplete but symmetric among all agents), leaving room for further model development.

[2] Focusing on book reviews and book sales on Amazon.com, Reimers and Waldfogel (2021) report that crowd ratings on the platform have a ten times larger impact on consumer welfare, compared to those provided by professional reviewers. As online crowd ratings are relatively cheap to manipulate (which is frequently done through anonymous accounts) compared to the cost of manipulating professional reviews, Reimers and Waldfogel (2021) findings further raise concerns about the role of manipulated reviews under differentiated-product oligopoly competition.

no competition in the review-manipulation stage even though there exist competitors in the market. Furthermore, as pointed out by Dellarocas (2006) in a monopoly setting, the review-manipulation in the oligopoly market has a signaling role, resulting in the improvement in rational consumer's expected surplus with manipulated reviews.

In the latter half of this article, we explore the behavioral analyses of manipulatied reviews with a representative naive consumer. Our motivation is as follows: A consumer may naively believe that observed (and publicly displayed) ratings are genuine and does not fully perceive the existence of manipulated reviews. If the market consists of such a naive representative consumer, a firm could exploit manipulated reviews to extract more profit. Particularly, the effect of manipulated reviews on a representative naive consumer's potential welfare loss would be of interest to market authorities and policymakers. In addition, to the best of our knowledge, welfare analyses of the manipulated reviews in varying-demand Bertrand and Cournot competitions with naive consumers have not been investigated in the literature.

In our behavioral model, a naive consumer cannot comprehend firms' review-manipulating activities, and they naively believe that an observed rating genuinely comes from the true product quality distribution without strategic manipulation. One notable advantage in our definition of a behavioral consumer is its nesting property: The proposed representative naive consumer nests that of the fully rational consumer as an extreme case, as well as nesting the fully naive consumer as another extreme case, enabling researchers to pursue comparative statics, notably welfare comparisons.

Under the behavioral model setting, review manipulation has two new effects: (i) an exploitation effect to extract surplus from a representative naive consumer and (ii) a business-stealing effect from the other firms. The first effect simply comes from biases caused by the manipulated reviews. The consumer purchases over-rated products excessively. The second effect arises as a naive consumer is attracted by a product with inflated rating with manipulated reviews. Then, strategic substitutability across review-manipulating actions emerges in contrast to the case with a rational consumer.

Because the only difference between a rational and naive consumer is the beliefs they form from an observed rating, the rational and naive consumers' surpluses are comparable. We numerically report the welfare implications of consumer naivete. Due to the naive consumer exploitation role, naive consumers experience large welfare losses compared to rational consumers.

Regarding the numerical analyses, we also quantify the amounts manipulated reviews generated under various competition measures, such as varying degrees of product substitutability. With rational consumers, we report that severe competition diminishes the amount of manipulated reviews, because harsh competition decreases each firm's profit and marginal gain from costly review manipulation also shrinks. However, with purely naive consumers, we have a different story. Severe competition can increase the amounts of manipulated reviews, notably because the excessive purchases by the biased consumer and high product substitutability can increase the marginal gain from review manipulation.

The remaining sections are organized as follows: Section 2 reviews the literature related to reviews and review manipulation. Section 3 explains our main review-manipulating oligopoly model with rational consumers. In this section, we mainly focus on a Bertrand duopoly competition model for expositional simplicity. Section 4 reports our main results with rational consumers, such as equilibrium properties and welfare analyses. Section 5 extends the model with behavioral consumers, who naively believe observed review ratings are genuine and thus cannot perceive firms' review-manipulating activities. In Section 5, we numerically analyze the welfare implications of review manipulation, and we assess welfare loss among naive consumers. This section also reports the amount of manipulated reviews generated by various competition measures. Section 6 extends the model in a few directions: firms with asymmetric priors of quality distributions, quantity competition, and an n-firm oligopoly. Lastly, Section 7 summarizes and concludes this study. Proofs and supplemental materials are reported in the Appendix sections.

2 Literature Review

Our study relates to several fields of literature. First, this article is related to the reputation studies. Specifically, our linear review rating model stems from the linear output framework proposed by Holmström (1999), which has recently and actively been applied to the design of dynamic rating systems with a single agent (Bonatti and Cisternas (2020) and Hörner and Lambert (2021)).[3]

Second, our paper is related to the theoretical literature on the properties of promotional reviews, including manipulated (fake) reviews. Under a monopoly market structure, broader aspects of manipulated reviews have been analyzed. Glazer et al. (2021) and Yasui (2020) analyze the dynamic roles of the platform dealing with fake reviews, and Aköz et al. (2020) study the potential signaling role of the monopolist's price combined with fake reviews. Regarding oligopoly, even though consumers use ratings to compare a product with others, and manipulated reviews are used to distinguish their product from competitors', the literature only occasionally studies the interaction between competition and review-manipulating be-

[3]See the survey of Bar-Isaac and Tadelis (2008) for the reputation literature. In addition, the rating design is also closely related to the literature of certification design, which was pioneered by Lizzeri (1999), currently an active research area. Closely related to our study, Zapechelnyuk (2020) reports that under some standard moral-hazard model settings, a simple binary quality certification is optimal.

havior. Under focused oligopoly competition settings, the early pioneering studies of Mayzlin (2006) and Dellarocas (2006) establish a new research field and analyze whether promotional reviews by sellers can benefit consumers. We contribute to this literature by analyzing review manipulation by multiple firms, which subsequently compete in a traditional Bertrand (or Cournot) competition market. This study also departs from the existing literature by introducing naive consumers and by reporting behavioral welfare implications.[4]

Third, this study also relates to the literature of advertisement with hidden types. The seminal studies of Nelson (1970) and Nelson (1974) introduce the concept of costly promotional activities conveying product quality information. Milgrom and Roberts (1986) formalize that, in a separating equilibrium, consumers rationally infer the seller's type after observing the advertising effort levels. In line with this argument, advertisements are often assumed to be perfectly observable, and models with imperfectly observed advertisements are relatively limited. Focusing on the duopoly market with a unit demand, Grunewald and Kräkel (2017) apply the Holmström (1999) model to imperfectly observed advertisements for vertically differentiated products.[5] The model in our study can be interpreted as the imperfectly observed advertisement in a flexible oligopoly model with vertically and horizontally differentiated products. Our contribution to this literature is that we reconcile two opposing views about the signaling role of the advertisement. While Nelson (1970) and Nelson (1974) argue that high-quality type provides more ads, Schmalensee (1978) argues that low-quality type provides more.In the equilibrium of our model, *ex-post* high-quality type provides more promotional activity while *ex-ante* weak seller (with low-quality in expectation) might put

[4]The empirical literature provides evidence of the existence and impact of review manipulation. Mayzlin et al. (2014) exploit a gap in the review process of two online accommodation platforms, indicating the existence of review manipulation. He et al. (2022a) combine data of fake-review offers on a social network platform and product sales rankings on Amazon.com to report a positive causal effect of fake reviews on Amazon sales, as well as pricing. In addition, He et al. (2022b) propose machine-learning methods to detect fake reviews, based on directly observed paid review manipulation activities. Yoshimoto and Zapechelnyuk (2023) report a dynamic review manipulation model of a monopoly seller, and test theoretical predictions by comparing restaurant reviews provided by online reviewers and professional guidebook reviewers. Using detected fake review data, Gandhi and Hollenbeck (2023) model consumer beliefs related to fake reviews, and then demonstrate structural estimations, reporting the intricate welfare consequences caused by fake reviews. Here, other relevant studies on the topics of ratings and reputation manipulation should be credited. An incomplete list of such studies is as follows: Chevalier and Mayzlin (2006) is a representative study of online reviews, investigating the effect of online book reviews on sales. Ratings and reviews are also closely related to dynamic reputation. With a dynamic model, Campbell et al. (2017) carry out a theoretical study on the relation of word-of-mouth and advertisement activities. Chevalier et al. (2018) study the dynamic response to reviews by managers. Hollenbeck (2018) provides an analysis of online reputation mechanisms, focusing on value chains.

[5]Some recent field experiments suggest that fake reviews and advertisements have distinctively different roles. Sahni and Nair (2020) show that consumers called restaurants more when the listing is revealed as "paid-advertisements" rather than non-paid listings. The study of Akesson et al. (2023) shows that consumers tend to be attracted to low-quality products if their reputation is inflated by fake reviews.

more promotional efforts.

Fourth, our study is related to behavioral economics literature, in which economic agents exhibit restricted conjecturing processes. Eyster and Rabin (2005) establish the epistemic game foundation with economic agents who do not fully consider other people's information. In this context, Kartik et al. (2007) study strategic communication, Murooka and Yamashita (2025) investigate an optimal trading mechanism in the presence of adverse selection, and Yasui (2020) applies the framework to a continuous-time monopoly market. However, to the best of our knowledge, an extension to oligopoly markets of this actively researched literature has not been reported, warranting a careful investigation. Our study bridges this behavioral literature and the classical oligopoly competition models, thus enabling researchers and practitioners to assess policy implications of competition in reputation manipulation against the behavioral consumer.

3 Model

In this section, we introduce our main model with rational consumers in a differentiated product oligopoly market, in which each firm produces one brand of products.[6] The game consists of five stages: (1) the drawing of hidden product quality types, (2) review manipulation (fake review writing actions), (3) the realization of product ratings with rating noises, (4) price (or quantity) choices, and finally, (5) consumption choices. After introducing some basic model background and notations, each of these stages is explained. Our basement is an *ex-ante* asymmetric Bertrand duopoly model with a representative rational consumer, yet it is flexible, allowing us to expand our model to Cournot competition and n-firm oligopoly.[7]

Firm i's profit ($i \in \{1, 2\}$) is defined as

$$\pi_i = (p_i - c_i)q_i - \frac{\phi_i}{2}F_i^2,$$

where p_i, q_i and F_i are firm i's price, quantity, and manipulation (fake-review) effort level, respectively. In addition, c_i and ϕ_i are costs for producing a unit of its product and a cost parameter for manipulation, adaptable to become asymmetric and heterogeneous.[8]

[6] We interchangeably use the terms "the representative consumer" and "consumers", as all are characterized by a quasi-linear utility function.

[7] In Section 6.2, we introduce a Cournot competition version of our model. In Section 6.3, we analyze n-firm oligopoly model. In Section 6.1, we provide our asymmetric model with a flexible interpretation regarding whether a high-quality or a low-quality seller provides more manipulated reviews, which could be viewed as imperfectly observed advertisements (see Nelson (1970), Nelson (1974), and Schmalensee (1978)).

[8] Following the precedent of Mayzlin et al. (2014), as well as traditional informative advertisement literature, the review manipulation (fake review writing) cost parameters (ϕ_is) and marginal production costs (c_is)

Given the qualities of each product, θ_1 and θ_2, we define the *ex-post* utility function for a representative rational consumer, as in Dixit (1979):

$$U = \theta_1 q_1 + \theta_2 q_2 - \frac{b_1}{2} q_1^2 - \frac{b_2}{2} q_2^2 - s q_1 q_2 - p_1 q_1 - p_2 q_2, \tag{1}$$

where b_1 and b_2, and s are constants, representing slope and substitution parameters. We suppose $b_1 b_2 - s^2 > 0$ for concavity of the utility function. We assume that c_i, ϕ_i, and b_i of firm $i \in \{1, 2\}$, as well as s, are common knowledge among all market participants.

In the equilibrium analysis in the subsequent sections, this *ex-post* utility function specifies the linear differentiated product demand system, notably with the expected qualities in its intercepts. More specifically, at the beginning of Stage 5, the consumer observes a publicly posted rating of each product, denoted as R_i, as a noisy signal of brand quality, although they cannot directly observe the underlying quality θ_i. In other words, the products are experience goods, so the true quality is revealed only after the purchase of the product. Given sellers' prices and ratings, at Stage 5, the representative rational consumer chooses the quantities to purchase to maximize *interim* expected utility, as described in the next section. Having described the last stage, we now outline the earlier stages in order.

At Stage 1, each firm draws its quality type θ_i from a normal distribution with its mean μ_i and variance $\sigma_{\theta,i}^2$, that is:

$$\theta_i \sim \mathcal{N}(\mu_i, \sigma_{\theta,i}^2).$$

The distributions of θ_is are independent of each other. Although the distributions of θ_is are common knowledge, the drawn type is private hidden information for each firm. We assume $\mu_i > c_i$ for all i.

At Stage 2, firm i makes the inflating manipulation (fake-review writing) effort, F_i.[9] At this stage, each of the two firms simultaneously decides to put some effort into manipulating its reviews, notably without knowing the opponent firm's hidden type.[10]

At Stage 3, the information on each product is gathered, and ratings for each product are revealed to the public. The rating for firm i's product is denoted as R_i and defined by

are public information. In addition, review manipulation cost parameters are heterogeneous. For example, some firms (e.g., large and brand-affiliated) could have higher costs, as, when revealed, review manipulation (fake reviews) damages their brand-wide reputation, while small and independent firms may have relatively smaller costs as their exits and entries are less costly. Note that, when ϕ_i goes to infinity, the cost of review manipulations becomes prohibitively high, and the payoff function is qualitatively equivalent to that of standard price (or quantity) competition with differentiated products.

[9] In the analysis, we will focus on linear strategy equilibrium, where the review manipulating effort is a linear function of each firm's type.

[10] This setting follows the tradition of a standard auction model, in which a bidder submits a bid without knowing the opponent bidders' valuations (or signals).

following Holmström (1999), as in Dellarocas (2006)[11]

$$R_i = \theta_i + F_i + \epsilon_i, \tag{2}$$

where ϵ_i is an i.i.d. random *review noise* drawn from a normal distribution with its mean zero and variance $\sigma^2_{\epsilon,i}$, that is:

$$\epsilon_i \sim \mathcal{N}(0, \sigma^2_{\epsilon,i}).$$

Intuitively, if the product quality is high, genuine customer reviews tend to be good, even though there will be some fluctuation due to taste heterogeneity of reviewers. A rating calculated from such genuine reviews is captured by $(\theta_i + \epsilon_i)$ in Equation (2). However, firm i can inflate its rating R_i by review manipulation, summarized as F_i in Equation (2).

Notably, the realization of ϵ_i and R_i in Stage 3 comes after the choice of F_i in Stage 2. In the context of online ratings, it is interpreted that, upon its entry, the seller distributes (unincentivized) product samples to consumers and, in addition, writes manipulated (fake) reviews. The sample-experienced consumers write genuine reviews $(\theta_i + \epsilon_i)$, which may or may not result in a high rating (R_i). On the other hand, manipulated reviews (F_i) are provided by hired fake reviewers (or in-house fake reviewers), who are incentivized to provide high ratings, although it becomes increasingly costly to compensate them, as represented by the convex manipulation cost function.[12]

At Stage 4, given the publicly available revealed ratings (R_is), each firm chooses its own price, $p_i \in [0, \infty)$ to maximize its profit.

Lastly, at Stage 5, the representative consumer chooses consumption quantities q_1 and q_2, and the market clears. The demand function is derived from the representative consumer's *interim* expected utility maximization, conditional on the observed ratings of products (R_is) and prices (p_is). Note that, before their consumption choices in Stage 5, the consumer can observe R_i for each firm, but not the realization of F_i or ϵ_i, so they cannot precisely disentangle the exact value of quality θ_i. Rather, consumers form conditional expectations,

[11] See also the survey of Bergemann and Bonatti (2019). See also the survey of Bergemann and Bonatti (2019). In addition, potentially negative θ_i and R_i could be questioned due to the assumption of normal distributions (see Hurkens (2014) for example), as well as a potentially negative manipulated (fake) review action F_i, which is linear in θ_i at equilibrium. In fact, this negativity in the review-manipulating action is not a nuisance in this study. We can replace a normal distribution with an elliptical distribution (Cambanis et al., 1981; Gómez et al., 2003; Ball, 2025) in a non-negative domain. See the Appendix for details.

[12] Alternatively, in the fashion of traditional non-online reviews, such as newspaper and consumer product magazine reviews, Equation (2) could also be interpreted as a combination of bribed and non-bribed professional reviews. For instance, if a new restaurant opens, the management may offer bribes to some but not all professional reviewers (e.g., local newspaper writers) to write favorable review articles for the restaurant, which is represented by the costly review manipulation activity, F_i. On the other hand, in general, it is not possible to bribe all reviewers: Other non-bribed reviewers (e.g., town magazine editors) honestly report genuine evaluations $(\theta_i + \epsilon_i)$, but their honest reviews come with their taste heterogeneity (ϵ_i).

as analyzed in the next section.

4 Equilibrium

In this article, we focus on a perfect Bayesian equilibrium where the seller's manipulation strategy is linear in its hidden type θ_i where $i \in \{1, 2\}$ is a firm index. To express the idea formally and concisely, we introduce a few vector notations: $\mathbf{p} = (p_1, p_2)$, $\mathbf{q} = (q_1, q_2)$, and $\mathbf{R} = (R_1, R_2)$. In addition, to analyze welfare, we also consider *ex-ante* expectation, variance, and covariance, with the notations of $E[\cdot]$, $Var[\cdot]$, and $Cov[\cdot, \cdot]$ in which integrals are calculated over all stochastic variables (θ_1, θ_2, ϵ_1, and ϵ_2) before the game starts (i.e., before Stage 1), although we often omit the term of "ex-ante" when the meaning is clear. By construction, ex-ante expectations, variances, and covariances are unconditional. The equilibrium is defined as follows.

Definition of Equilibrium Manipulated review equilibrium is characterized by the following conditions:

1. *Expected utility maximization*:

$$(q_1(\mathbf{p}; \mathbf{R}), q_2(\mathbf{p}; \mathbf{R})) = \arg\max_{q_1, q_2} E_c[U|\mathbf{R}, \mathbf{p}];$$

2. *Profit-maximizing pricing*: Given $p_{-i} = p^*_{-i}(\mathbf{R})$,

$$p^*_i(\mathbf{R}) = p_i(p_{-i}, \theta_i, \mathbf{R}) \equiv \arg\max_{p_i} E_i[(p_i - c_i)q_i(\mathbf{p}; \mathbf{R})) - \frac{\phi_i}{2} F_i^2 | \theta_i, \mathbf{R}];$$

3. *Profit-maximizing reputation manipulation strategy*: Let $q^*_i(\mathbf{R}) = q_i(p^*_1(\mathbf{R}), p^*_2(\mathbf{R}); \mathbf{R})$. Then,

$$F^*_i(\theta_i) = F_i(\theta_i; F^*_{-i}(\cdot)) \equiv \arg\max_{F_i} E_i[(p^*_i(\mathbf{R}) - c_i)q^*_i(\mathbf{R}) - \frac{\phi_i}{2} F_i^2 | \theta_i];$$

4. *Linear reputation manipulation strategy*: For some constant α_i and γ_i

$$F^*_i(\theta_i) = \alpha_i \theta_i + \gamma_i;$$

5. *Passive belief*: For any p_1 and p_2,

$$E_c[\theta_i|\mathbf{R}, p_1, p_2] = E_c[\theta_i|\mathbf{R}, p^*_1(\mathbf{R}), p^*_2(\mathbf{R})] \,(= E_c[\theta_i|\mathbf{R}]) \,.$$

Related to the second condition, at any equilibrium, the profit-maximizing price does not vary with own quality θ_i, and does not convey any information in addition to the rating. This is an immediate property derived from the model assumption, rather than an equilibrium restriction. Recall that, at the pricing stage, θ_i does not appear in the profit function, and all other components in the formula are already realized. Therefore, there is no single-crossing property at the pricing stage.[13] Thus, on the equilibrium path, the consumer does not infer the underlying quality from the prices. The last condition is imposed to complete the analysis *off* the equilibrium-path, which characterizes the shape of the residual demand for each product. Because the consumer does not extract any additional information of the quality from the price at equilibrium, it is natural to suppose that they cannot do that off-equilibrium.[14] Then, the consumer's belief is characterized enough to solve the model (almost) backwardly.

At the beginning of Stage 5, conditional on the observed ratings and prices, the *interim* expected utility of the representative consumer is defined as:

$$E_c\left[U|\mathbf{R},\mathbf{p}\right] = E_c\left[\theta_1|\mathbf{R},\mathbf{p}\right]q_1 + E_c\left[\theta_2|\mathbf{R},\mathbf{p}\right]q_2 - \frac{b_1}{2}q_1^2 - \frac{b_2}{2}q_2^2 - sq_1q_2 - p_1q_1 - p_2q_2. \tag{3}$$

That is, the *interim* expected utility is deterministic, except for each product's quality. Specifically, for firm $i \in \{1,2\}$, the rational consumer forms the conditional expectation of the true product quality θ_i as[15]

$$E_c\left[\theta_i|\mathbf{R},\mathbf{p}\right] = E_c\left[\theta_i|R_i\right] = \mu + \frac{Cov_c\left[\theta_i,R_i\right]}{Var_c\left[R_i\right]}(R_i - E_c[R_i]) \equiv \underbrace{Y_i}_{\text{firm } i\text{'s reputation}}. \tag{4}$$

To promote intuitive understanding, we introduce the shorthand notation of Y_i and broadly call it firm i's (rational) *reputation*. Note that, the first equality holds because the prices

[13] More explicitly, suppose that there exists an equilibrium where the prices signal the underlying quality, i.e., $p_i^*\left(\mathbf{R},\theta_i\right) \neq p_i^*\left(\mathbf{R},\widetilde{\theta}_i\right)$ for some θ_i and $\widetilde{\theta}_i$ such that $\theta_i \neq \widetilde{\theta}_i$. Still, the consumer's utility maximization gives the demand for product i as in the first condition of the equilibrium: $q_i\left(\mathbf{p},\mathbf{R}\right)$. Then, firm i with θ_i and $\widetilde{\theta}_i$ faces the same objective function at the pricing stage. This results in the same profit maximizing price, $p_i\left(p_{-i},\theta_i,\mathbf{R}\right) = p_i\left(p_{-i},\widetilde{\theta}_i,\mathbf{R}\right)$. Thus, the price at equilibrium does not depend on the underlying quality θ_i. This property depends on the timing of the game. See Aköz et al. (2020) for a model with pricing before the rating realization and a signaling role of the prices.

[14] Dellarocas (2006) imposed the same condition for the off-equilibrium belief implicitly. Bonatti and Cisternas (2020) imposed *limit sensitivity* condition to pin down off-equilibrium response in a continuous time model.

[15] See the Appendix Section A.1 for the derivation steps, in which we use the linear conditional expectation property of a joint normal distribution (DeGroot, 2005). This linear conditional expectation property also holds with a joint elliptical distribution, notably with a nonnegative domain (Cambanis et al., 1981; Gómez et al., 2003; Ball, 2025).

convey no additional information on top of the ratings, and the other product's rating R_j of firm $j \neq i$ does not depend on θ_i (see Equation (2)). Thus, the consumer's expectation of each product's quality is calculated by the projection of θ_i only on R_i.

Moreover, because the consumer takes firm i's review manipulation strategy $F_i(\theta_i) = \alpha_i\theta_i + \gamma_i$ as given, the consumer evaluates the equilibrium rating as $R_i = \theta_i + F_i(\theta_i) + \epsilon_i = (1+\alpha_i)\theta_i + \gamma_i + \epsilon_i$. Then, as θ_i and ϵ_i are i.i.d., and the consumer knows it, the ratio of unconditional variance-covariance is written as

$$\frac{Cov_c\left[\theta_i, R_i\right]}{Var_c\left[R_i\right]} = \frac{(1+\alpha_i)\,\sigma_{\theta,i}^2}{(1+\alpha_i)^2\,\sigma_{\theta,i}^2 + \sigma_{\epsilon,i}^2} = \frac{(1+\alpha_i)}{(1+\alpha_i)^2 + (\sigma_{\theta,i}/\sigma_{\epsilon,i})^{-2}} \equiv \underbrace{\lambda_i}_{\text{curving coefficient for firm } i}, \tag{5}$$

and we label λ_i as the consumer's *curving coefficient* for firm i (or, for a rating of firm i's product). In addition, the consumer's unconditional expectation of firm i's rating is

$$E_c[R_i] = E_c[\theta_i] + E_c[F_i] = (1+\alpha_i)\,\mu + \gamma_i. \tag{6}$$

In Equation (5), the quotient term of $\sigma_\theta/\sigma_\epsilon$ is a *signal-to-noise ratio*, representing the accuracy of the rating system without review manipulation (i.e., without fake reviews). The curving coefficient λ_i is increasing in this ratio. The intuition is that, when this ratio is large, the consumer finds that the review rating (R_i) is relatively more informative about true product quality. Thus, the reputation Y_i becomes more sensitive to the rating R_i.

By arranging Equations (4), (5), and (6), firm i's reputaton among the rational consumers at the beginning of Stage 5 can be intuited as

$$\underbrace{Y_i}_{E_c[\theta_i|R_i]} = \mu + \lambda_i \left(\underbrace{\theta_i + F_i + \epsilon_i}_{R_i} - \underbrace{\left\{ E_c[\theta_i] + E_c[F_i] \right\}}_{E_c[R_i]} \right), \tag{7}$$

where reputation Y_i and manipulation effort F_i have a linear relationship. Equation (7) also manifests the rational consumer's curving process: In equilibrium, the rational consumer deduces their conditional expectation of product quality θ_i, notably by filtering the manipulated (and noisy) rating R_i through the *curving coefficient* λ_i.[16]

At Stage 5, given the reputations, $Y_1 = E_c\left[\theta_1|R_1\right]$ and $Y_2 = E_c\left[\theta_2|R_2\right]$, the first-order conditions for the consumer's *interim* utility maximization in Equation (3) with respect to

[16] In addition, regarding the ex-ante (before Stage 1) expectation of reputation, $E[Y_i] = \mu$ holds at equilibrium as long as the consumers are rational.

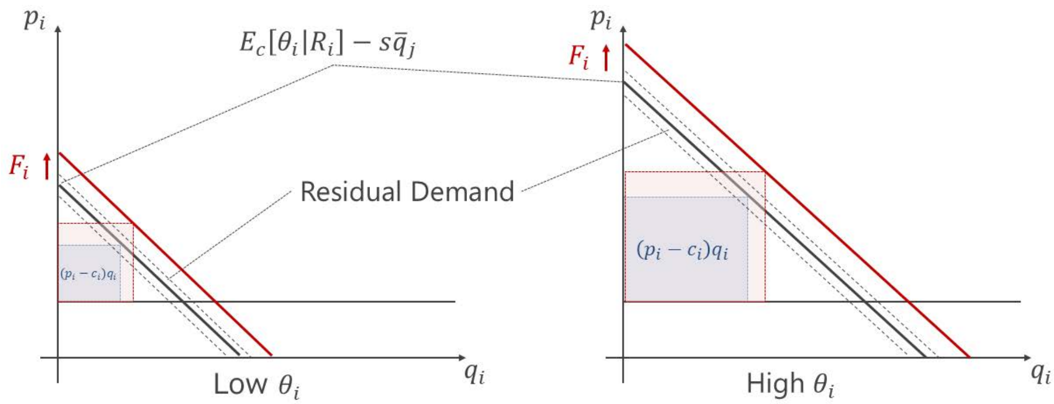

Figure 1: Graphical Explanations of Reputation ($Y_i = E_c[\theta_i|R_i]$) and Positive α_i

q_1 and q_2 are:

$$\begin{aligned} p_1 &= Y_1 - b_1 q_1 - s q_2 \\ p_2 &= Y_2 - b_2 q_2 - s q_1, \end{aligned} \tag{8}$$

and we obtain a linear inverse demand system of Equation (8),[17] where the constant terms are determined by the reputation of each product, $Y_i = E_c[\theta_i|R_i]$. Note that, after ratings are publicly announced at Stage 3, the reputations are constant for firms at Stage 4, as consumers do not infer hidden quality from prices. Figure 1 depicts the review manipulation incentive of firm i based on Equation (7) and (8), in which the reputation term is a part of the intercept of the inverse demand function. Because the reputation ($Y_i = E_c[\theta_i|R_i]$) and manipulated reviews (F_i) have a linear relationship in Equation (7), given the consumers' belief, when firm i increases F_i, it linearly pushes up the intercept of the inverse demand function, resulting in a larger markup (or revenue).

At Stage 4, as an equilibrium outcome of price competition with a linear demand system, firm 1's profit function becomes quadratic in its equilibrium quantity q_i, which is then linear in its reputation ($Y_1 = E_c[\theta_1|R_1]$). Specifically, firm 1's profit function is reduced to the following:

$$\pi_1 = \frac{1}{\beta_i}(q_i^*)^2 - \frac{\phi_1}{2}F_1^2, \tag{9}$$

where

$$\beta_i = b_j/(b_1 b_2 - s^2)$$

[17]Even if the actual choices consumers face are discrete, the decisions across the whole consumers can be aggregated into a smooth demand function system derived from a representative consumer's decision-making. See Armstrong and Vickers (2015), for the details of aggregating condition.

is a slope of the demand for product i. The Bertrand outcome q_i^* is further decomposed as

$$q_i^* = L_i (Y_i - c_i) - M_i (Y_j - c_j) \tag{10}$$

where $L_i = \frac{b_j(2b_1b_2 - s^2)}{(4b_1b_2 - s^2)(b_1b_2 - s^2)}$ and $M_i = \frac{b_1b_2s}{(4b_1b_2 - s^2)(b_1b_2 - s^2)}$.[18] Intuitively, as the market becomes more competitive (with s close to b_i), the equilibrium quantity becomes more sensitive to own and other's reputation (higher L_i and M_i), and other's reputation becomes relatively more important (higher M_i/L_i).

At Stage 3, the rating R_i is realized with the review shock ϵ_i. Thus, at Stage 2, when firms simultaneously set their manipulation levels F_i, they are uncertain about the realization of their respective reputations, $Y_i = E_c[\theta_i|R_i]$. Therefore, the profit function for firm 1 is evaluated with an expectation conditional on its own product's quality

$$E[\pi_i|\theta_i, F_i] = \beta_i^{-1} E\left[(q_i^*)^2 |\theta_i, F_i\right] - \frac{\phi_i}{2} F_i^2 \tag{11}$$

$$= \beta_i^{-1} \left\{ Var(q_i^*|\theta_i, F_i) + E[q_i^*|\theta_i, F_i]^2 \right\} - \frac{\phi_i}{2} F_i^2 \tag{12}$$

Here, the manipulation of firm i does not affect the conditional variance of the equilibrium quantity.

Lemma 1. *$\partial Var(q_i^*|\theta_i, F_i) / \partial F_i = 0$.*

Proof. By Equations (7) and (10), q_i^* is linear in $\theta_i, \theta_j, F_i, F_j, \epsilon_i$ and ϵ_j. Furthermore, from the viewpoint of firm i, F_j is given by $F_j = F_j^*(\theta_j) = \alpha_j\theta_j + \gamma_j$. Thus, given θ_i and F_i, the remaining sources of variations in q_i^* are θ_j, ϵ_j $(j \neq i)$ and ϵ_i, which are uncorrelated with θ_i and additive separable with (θ_i, F_i) in the expression of q_i^*. Therefore, F_i does not change the variation of q_i^* given θ_i and F_i. □

Then, the first-order condition w.r.t. F_i is written as

$$0 = 2\beta_i^{-1} \frac{\partial E[q_i^*|\theta_i, F_i]}{\partial F_i} E[q_i^*|\theta_i, F_i] - \phi_i F_i$$

Here, recall that $q_i^* = L_i (Y_i - c_i) - M_i (Y_j - c_j)$ and $E[Y_j|\theta_i, F_i] = E[Y_j]$, and note $\frac{\partial E[q_i^*|\theta_i, F_i]}{\partial F_i} = \frac{\partial E[q_i^*|\theta_i, F_i]}{\partial Y_i} \frac{\partial Y_i}{\partial F_i} = L_i\lambda_i$. Then, the marginal gain from the manipulation is decomposed into

[18] See Appendix C for the derivation. Furthermore, by replacing L_i and M_i, we can obtain a similar quadratic profit function for a firm in a Cournot quantity competition and apply the same logic. See also the Appendix C for details.

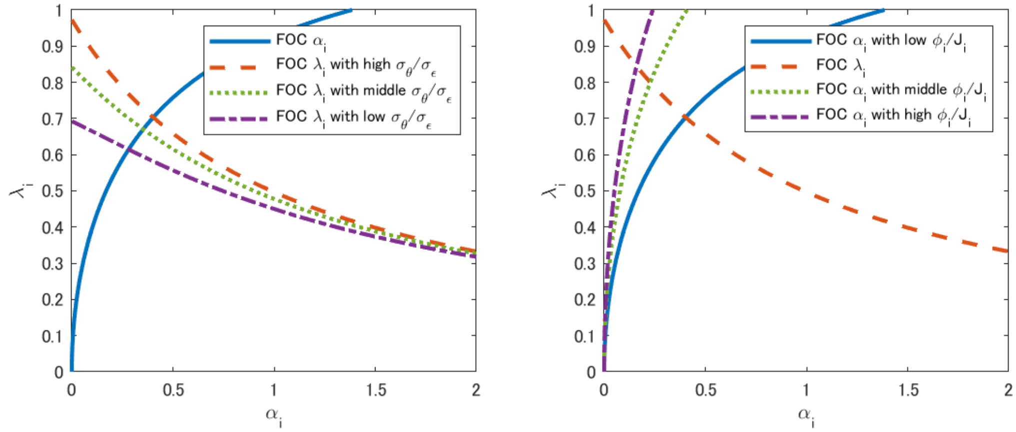

Figure 2: Graphical Explanation of Equations (16) and (17)

two parts regarding whether it depends on the hidden information of the firm i (θ_i, F_i):

$$\begin{aligned} 0 &= 2\beta_i^{-1} L_i \lambda_i E\left[L_i\left(Y_i - c_i\right) - M_i\left(Y_j - c_j\right) |\theta_i, F_i\right] - \phi_i F_i \\ &= 2\beta_i^{-1} L_i \lambda_i \left(E\left[q_i^*\right] + L_i\left(E\left[Y_i|\theta_i, F_i\right] - E\left[Y_i\right]\right)\right) - \phi_i F_i \quad (13) \\ &= 2\beta_i^{-1} L_i \lambda_i \left(E\left[q_i^*\right] + L_i \lambda_i\left(\theta_i - E\left[\theta_i\right] + F_i - E\left[F_i\right]\right)\right) - \phi_i F_i \quad (14) \end{aligned}$$

By rearranging it, the firm i' optimal manipulation strategy F_i satisfies

$$\left(\phi_i - 2\beta_i^{-1} L_i^2 \lambda_i^2\right) F_i = 2\beta_i^{-1} L_i^2 \lambda_i^2 \theta_i + 2\beta_i^{-1} L_i \lambda_i \left(E\left[q_i^*\right] + L_i \lambda_i\left(-E\left[\theta_i\right] - E\left[F_i\right]\right)\right) \quad (15)$$

On the one hand, α_1 is determined by matching the slope coefficient in the above first-order condition with that in the linear strategy, $F_i(\theta_i) = \alpha_i \theta_i + \gamma_i$. On the other hand, the rational consumers' curving coefficient (λ_i) is reported in their inference process of Equation (5). Consequently, we have the system of

$$\alpha_i = \frac{2\beta_i^{-1} L_i^2 \lambda_i^2}{\phi_i - 2\beta_i^{-1} L_i^2 \lambda_i^2} = \frac{2\lambda_i^2}{\left(\beta_i^{-1} L_i^2/\phi_i\right)^{-1} - 2\lambda_i^2} \quad (16)$$

$$\lambda_i = \frac{(1+\alpha_i)}{(\sigma_{\theta,i}/\sigma_{\epsilon,i})^{-2} + (1+\alpha_i)^2}. \quad (17)$$

Regarding optimality, the second-order condition for the profit maximization is satisfied if and only if $\phi_i - 2\beta_i^{-1} L_i^2 \lambda_i^2 > 0$. That is, the review manipulation cost parameter (ϕ_i) is large enough relative to the marginal benefit related term of business-stealing effect $(\beta_i^{-1} L_i^2)$. Given this condition, $\alpha_i > 0$ holds for firm 1's profit-maximizing strategy. Furthermore, this condition also implies $\lambda_i > 0$ in the consumers' equilibrium curving process. Figure 4 illustrates the mutual interactions between Equations (16) and (17) with varying *signal-to-noise ratio* $(\sigma_{\theta,i}/\sigma_{\epsilon,i})$ and *benefit-cost ratio* $(\beta_i^{-1}L_i^2/\phi_1)$ values, numerically exemplifying the equilibrium relationship between α_i and λ_i.[19] $E[F_i]$ is characterized by the expectation of the equation (14).

$$\begin{aligned}0 &= 2\beta_i^{-1}L_i\lambda_i\left(E\left[q_i^*\right]+L_i\lambda_i\left(E\left[\theta_i\right]-E\left[\theta_i\right]+E\left[F_i\right]-E\left[F_i\right]\right)\right)-\phi_iE\left[F_i\right]\\ &= 2\beta_i^{-1}L_i\lambda_iE\left[q_i^*\right]-\phi_iE\left[F_i\right]\end{aligned}$$

If the consumer is rational (i.e., $E_c[\theta_i|R_i] = E[\theta_i|R_i]$), then $E[Y_i] = E[\theta_i]$ holds and $E[F_i]$ is explicitly characterized as

$$E[F_i] = \frac{2\beta_i^{-1}L_i^2\lambda_i}{\phi_i}\left(\left(E\left[\theta_i\right]-c_i\right)-\frac{M_i}{L_i}\left(E\left[\theta_j\right]-c_j\right)\right) \tag{18}$$

For completeness of the equilibrium characterization, γ_i is characterized by

$$\gamma_i = E\left[F_i\right]-\alpha_iE\left[\theta_i\right]. \tag{19}$$

4.1 Equilibrium Properties

First, we can observe that Equations (16)-(19) do not depend on firm 2's strategy, $F_2(\cdot)$. Thus, if another firm is taking the equilibrium manipulation strategy $F_2^*(\cdot)$ and the consumer has a rational belief on it, firm 1's profit-maximizing manipulation strategy $F_1(\cdot)$ does not rely on the shape of $F_2^*(\cdot)$, resulting in the following proposition.[20]

Proposition 1 (Unresponsive Best Responses in Manipulated Reviews)**.** *If the consumer has a rational and passive belief that sellers implement linear strategies, seller i's profit-maximizing strategy is linear and does not depend on seller j's strategy ($j \neq i$).*

This seemingly paradoxical proposition can be intuitively understood as follows. Suppose that firm 2 finds manipulative behavior costly given a strict regulation by the government or the consumer's strict attitude against manipulative behavior (i.e., high ϕ_2). Firm 2 becomes (known to be) relatively truthful to the consumer and reduces the intensity of manipulation (low $E_c[F_2]$). The unresponsive property in the above proposition implies that such truthful behavior does not propagate firm 1's strategy, given the rational consumer's

[19] A larger $\beta_1^{-1}L_1^2/\phi_i$ roughly implies that the (marginal) benefit part of the conditional expected profit function is relatively more influential compared to the (marginal) cost part, based on the first-order condition of Equation (15).

[20] In contrast, if consumers are naive, a strategic substitutability property emerges among review-manipulating actions (see Proposition 8).

curving process. This might be surprising given that the reputation of firm 2 (Y_2) shifts the demand for firm 1's product. In fact, if firm 2 reduces the manipulation intensity, it decreases its own expected *rating* (low $E_c[R_2]$). However, as long as the consumer is rational and knows that the manipulation is costly for firm 2, the consumer can rationally expect that firm 2's manipulation is reduced (low $E_c[F_2]$). Therefore, the *reputation* of firm 2 remains the same on average (same $E_c[Y_2]$) even after the reduction of the manipulation. Thus, on average, firm 1's demand curve and incentive to manipulate its rating do not change.

Furthermore, signs of the coefficients for the equilibrium strategies are determined by observing the first-order conditions.

Proposition 2 (Signs of the Coefficients at Equilibrium)**.** *At any linear equilibrium with a passive consumer's belief, α_i, λ_i are positive.*

See Appendix B.2 for a proof. Intuition is explained as follows. As depicted in Figure 1, the manipulated review linearly raises the intercept of its own demand curve, and that then linearly increases the equilibrium quantity and price. Thus, the manipulated review quadratically increases the firm's variable profit from the market competition, as depicted in the rectangle areas in Figure 1. Furthermore, given the linear rating of Equation (2), if the seller privately observes high quality (high θ_i), then the seller expects a high reputation (i.e., high $Y_i = E_c[\theta_i|R_i]$) even without writing manipulated reviews (i.e, $F_i = 0$). Because of the quadratic feature of the profit function, the marginal effect of review manipulation on profit is higher when the seller obtains high quality. Thus, the seller with a high quality (the right-hand panel in Figure 1) has a greater incentive to produce manipulated reviews than the seller who gets a low quality (the left-hand panel in Figure 1).[21]

Given the equilibrium manipulation-strategy $F_i^*(\theta_i) = \alpha_i\theta_i + \gamma_i$ for any θ_i, other parts of the equilibrium can also be straightforwardly characterized. The existence and uniqueness of equilibrium hinges on the existence and uniqueness of α_i and γ_i characterized by Equations (16)-(19).

Proposition 3 (Existence and Uniqueness of the Equilibium)**.** *If $\sigma_{\theta,i}/\sigma_{\epsilon,i} > 1$, then the linear equilibrium with a passive consumer's belief always exists and it is unique.*

Graphically, in Figure 4, Equations (16) and (17) intersect at least once. By assuming the informative rating condition of $\sigma_{\theta,i}/\sigma_{\epsilon,i} > 1$, we obtain a monotonically decreasing graph of Equation (17), which uniquely intersects with a monotonically increasing graph of Equation (16). See Appendix A.2 for a formal proof.

[21]This signaling mechanism behind the positive α_i is in line with the study of Dellarocas (2006), which analyzes the online opinion forums without oligopolistic price (or quantity) choices.

Now, we perform comparative statics on the key (combinations of) parameters to the equilibrium variables. In the rest of the section, we focus on the parameter set where the unique equilibrium is guaranteed ($\sigma_{\theta,i}/\sigma_{\epsilon,i} > 1$). First, we report the behavior of the equilibrium coefficient α_i and curving coefficients λ_i, which will play key roles in the welfare analysis in Section 4.2.

Proposition 4. *At equilibrium,*

1. *α_i and λ_i do not vary with c_i, $E[\theta_i]$, ϕ_j, c_j, nor $E[\theta_j]$ ($j \neq i$),*
2. *α_i is increasing in $(L_i^2/\beta_i\phi_i)$ and λ_i is decreasing in $(L_i^2/\beta_i\phi_i)$,*
3. *α_i and λ_i are increasing in $(\sigma_{\theta,i}/\sigma_{\epsilon,i})$, and*
4. *$E[F_i]$ is increasing in $(\sigma_{\theta,i}/\sigma_{\epsilon,i})$ and $(E[\theta_i] - c_i)$, decreasing in ϕ_i and $(E[\theta_j] - c_j)$.*

The proposition is proved by observing Equations (16)-(18). See Figure 4 for a graphical explanation.

The first result shows that the prior quality level or production cost, and the other firm's manipulation cost do not alter α_i or λ_i. The second result shows that the effects of the slope of the inverse demand b, the substitution parameter s are captured by the slope of the demand function β_i and the marginal effect of reputation on the equilibrium quantity L_i.

Regarding regulatory design, the second and fourth results clarify the distinction between seemingly similar regulatory treatments: (i) the stringent censorship and severe punishment against the manipulated reviews (a higher ϕ_i) and (ii) improvement of authentic customer review accuracy (a smaller $\sigma_{\epsilon,i}$).

Intuitively, an increase in $(L_i^2/\beta_i\phi_i)$ implies a rise in the marginal impact on the seller's profit in the market competition stage (Stage 4) relative to the marginal cost of review manipulation (in Stage 2). Then, in equilibrium, as the amount of manipulated reviews (F_i) increases, the signaling property of manipulated reviews is enhanced (i.e., higher α_i). Taking the increased number of manipulated reviews, the consumers in the equilibrium curve the observed rating even more (i.e., smaller λ_i).

By contrast, a higher $(\sigma_{\theta,i}/\sigma_{\epsilon,i})$ implies a precise rating system even without manipulated reviews. Therefore, the consumers' purchasing behavior is highly responsive to the ratings of each product (higher λ_i). By taking this into account, the sellers in equilibrium create more manipulated reviews and the signal effect of manipulated reviews (higher F_i) is also enhanced (higher α_i).

Because the demand substitution parameter s appears in β_i, L_i, and M_i in Equations (16)-(18), a natural question emerges as to how the equilibrium variables behave as the market becomes competitive (with higher s). Figure 3 reports a numerical analysis on α_i, λ_i, and $E[F_i]$ with the substitution parameter s on the horizontal axis (left panels), along with the relevant analysis with ϕ_i on the horizontal axis (right panels).[22,23] In all panels, the blue line shows the equilibrium numerics with the representative rational consumer, while the dotted red line shows those of the representative naive consumer (which will be explained in Section 5). In this section, we focus on the interpretations of the representative rational consumer.

As the products become more substitutable (higher s), the expected amount of review manipulation ($E[F_i]$) increases in the bottom-left panel. Intuitively, high product substitutability shifts down residual demand and shrinks the (marginal) benefit from manipulating reviews.

By contrast, the slope of the linear review manipulation strategy (α_i) increases when s goes up (top-left panel). In other words, the ratings react sensitively to the underlying qualities when products are similar. Intuitively, as s goes up, the difference in qualities becomes more crucial and the mechanism behind Proposition 1 works strongly. Then, as reported in the middle-left panel, given high fluctuations of R_i caused by high α_i, the rational consumer curves ratings more (lower λ_i) so that the rational consumer can compatibly interpret the distribution of inflated R_i and the known distribution of θ_i.

The center column of Figure 3 illustrates results from Proposition 4 related to the manipulation cost parameter ϕ_i. As ϕ_i increases, both the slope of the linear review manipulation strategy (α_i in the top-center panel) and the expected amount of manipulated reviews ($E[F_i]$ in the bottom-center panel) decrease. This is because, regardless of the firm's type, when the marginal cost of manipulation becomes relatively high, manipulative activity becomes unattractive. Thus, both the variation (α_i) and the level ($E[F_i]$) of the manipulated reviews diminish. Then, because a lower α_i implies less fluctuation in R_i, the representative rational consumer does not curve the observed ratings largely (higher λ_i) as reported in the middle-center panel.

The right column of Figure 3 reports the effect of decreasing review noise variance $\sigma^2_{\epsilon,i}$.[24]

[22] See the Appendix D for the extreme cases of $s \to 0$ (no substitute) and $s \to b$ (perfect substitute).

[23] In Figure 3, b_i is fixed to 1, as is the maximum level of s.

[24] Regarding the i.i.d. review noise $\epsilon \sim \mathcal{N}(0, \sigma^2_{\epsilon,i})$ in Equation (2), the magnitude of its variance (i.e, $\sigma^2_{\epsilon,i}$) could be of interest to both platforms and market-regulating authorities. For example, during the aggregation of genuine reviewers' ratings for a product, a platform may decide whether to exclude (or include) the ratings of infrequent or inexperienced reviewers, or those who provide ratings but do not leave comments. In this way, a platform can tune the reviewer population, thus modifying the review noise variance. A market authority could encourage (or discourage) this review noise tuning process. In this vein, in this and subsequent sections, we will analyze the policy implications for rational and naive consumers' surpluses, notably with respect to the magnitude of review noise variance. With a higher signal-to-noise ratio, a rating conveys more accurate product-quality information. Then, the consumer reacts more sensitively to the ratings (i.e., higher

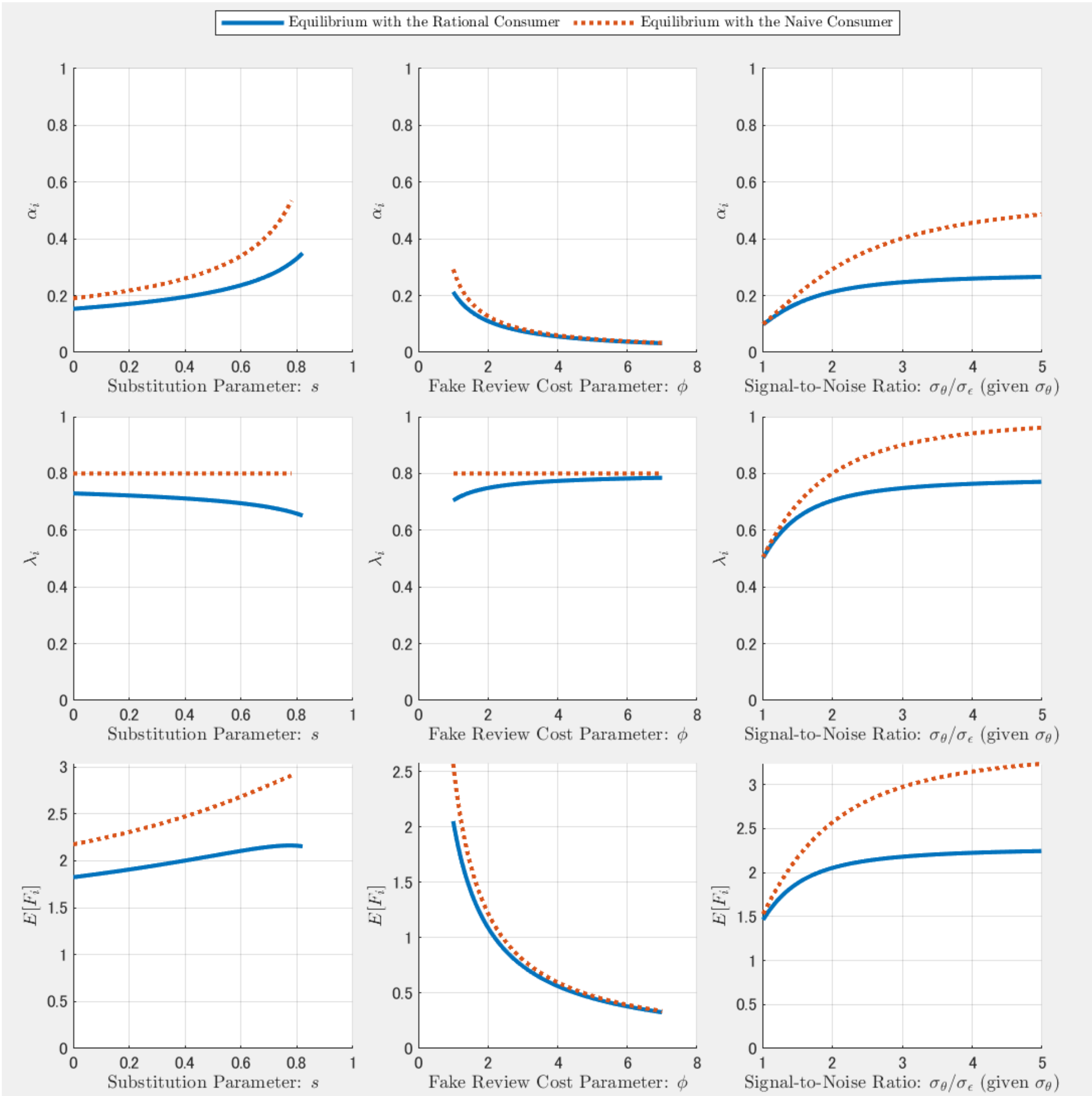

Figure 3: Comparative Statics - Equilibrium Coefficients

Next, to promote further understanding of the welfare analysis, we consider the properties of equilibrium prices and quantities, as well as variances of reputation terms (Y_i's). Because equilibrium quantities and prices are linear in the sellers' reputations (Y_i's) (see Equations (8) and (10)) and the rational consumer correctly curves the inflated ratings in expectation (i.e., $E[Y_i] = E[\theta_i]$), the ex-ante expected equilibrium prices and quantities are the same as ones without review manipulation as long as consumers are rational.

λ_i). Thus, the marginal gain of manipulative behavior increases, and then, both the level of manipulative behavior ($E[F_i]$) and its signaling effect (α_i) are enhanced.

Proposition 5. *If consumers are rational,* $E[p_i^*]$ *and* $E[q_i^*]$ *are the same as ones without manipulated reviews.*

It is worth noting that, by contrast, ex-ante variances of equilibrium prices and quantities are largely altered by manipulated reviews, which deviates from the previous literature.

Proposition 6. $Var[p_i^*]$ *and* $Var[q_i^*]$ *are larger than those without manipulated reviews.*

Proof. Because q_i^* and p_i^* are linear in reputations ($Y_i, i = 1, 2$) and the reputations are uncorrelated (i.e., $Cov[Y_1, Y_2] = 0$), the $Var[p_i^*]$ and $Var[q_i^*]$ are linear in the variances of the reputations ($Var[Y_i]$'s). By using Equation (7), we have the ex-ante variance of reputation:

$$\begin{aligned} Var\left[Y_i\right] &= \lambda_i^2 Var\left[R_i\right] \\ &= \left(\frac{(1+\alpha_i)\sigma_{\theta,i}^2}{(1+\alpha_i)^2\sigma_{\theta,i}^2+\sigma_{\epsilon,i}^2}\right)^2 Var\left[(1+\alpha_i)\theta_i+\epsilon_i\right] \\ &= \left(\frac{(1+\alpha_i)^2}{(1+\alpha_i)^2+(\sigma_{\theta,i}/\sigma_{\epsilon,i})^{-2}}\right)\sigma_{\theta,i}^2. \end{aligned} \tag{20}$$

As the equilibrium coefficient α_i is positive, $Var[Y_i]$ becomes larger with manipulated reviews than without manipulated reviews (i.e., the case of $\alpha_i = 0$ and $\gamma_i = 0$). Thus, $Var[q_i]$ and $Var[p_i]$ are larger with manipulated reviews than without manipulated reviews. □

These propositions result in some key welfare consequences of the manipulated reviews in the oligopoly market as described in the following Subsection.

4.2 Welfare Analysis

In this section, we analyze the effects of review manipulation on *ex-ante* surpluses "at equilibrium". At the core, the *ex-ante* rational consumer surplus is reduced to a function of moments of q_i's by inserting the inverse demand functions into the surplus:

$$E[U] = E\left[\frac{b_1}{2}q_1^2 + \frac{b_2}{2}q_2^2 + sq_1 \cdot q_2\right].$$

By replacing the second moments in the above equation with the variances and the first moments, we can decompose the surplus into the manipulation-variant and invariant parts:[25]

$$E[U] = \underbrace{\frac{b_1}{2}E\left[q_1\right]^2 + \frac{b_2}{2}E\left[q_2\right]^2 + s \cdot E\left[q_1\right]E\left[q_2\right]}_{\text{review-manipulation invariant part}} + \underbrace{\frac{b_1}{2}Var\left[q_1\right] + \frac{b_2}{2}Var\left[q_2\right] + s \cdot Cov\left[q_1, q_2\right]}_{\text{review-manipulation variant part} > 0} \tag{21}$$

With rational consumers, as manipulated reviews do not change the *ex-ante* expected reputations, prices, and quantities at equilibrium (Proposition 5), the first three terms in Equation (21) do not change. Thus, the following lemma holds.

Lemma 2. *Review manipulation affects the ex-ante consumer surplus only via the ex-ante variances and covariance of* (q_i)*s.*

Intuitively, the ex-ante variance in the consumption of each product (q_i) positively contributes to the ex-ante consumer surplus. This is because the varying consumption means that the rational consumer manages to extract some information from ratings, updates their beliefs on the product qualities, and then, adjusts their consumption levels accordingly.[26,27]

Furthermore, in contrast to previous studies, the covariance between q_1 and q_2 appears in the last term. In the extreme case of $s = 0$, the two products are not substitutable, the consumption of product i (q_i) depends only on the reputation of product i (Y_i). However, if the products are substitutable $(s > 0)$, the consumer adjusts their consumption level (q_i) not only based on one product's reputation (Y_i), but also according to the other product's reputation $(Y_j, j \neq i)$. For instance, even if Y_2 does not vary, high (low) Y_1 implies high (low) q_1 and low (high) q_2. Thus, even without the variation in Y_2, the variance in q_2 is caused by substitution,[28] which does not contribute to the consumer surplus, but is accounted for $(b_2/2)Var[q_2]$ in Equation (21), thus over-counted in terms of the consumer surplus. Therefore, the last term $(s \cdot Cov\left[q_1, q_2\right] < 0)$ exists to adjust such an over-count caused by

[25] This *ex-ante* consumer surplus decomposition applies to both Bertrand and Cournot competition, although the *ex-ante* expectations, variances, and covariances vary across different competition environments.

[26] In other words, as the signal (R_i) becomes more informative, the posterior-mean (Y_i) and the action of the receiver (q_i) distribute more. This argument is well known in the information design literature (e.g., Bergemann and Bonatti (2019)).

[27] See Dellarocas (2006) and Bonatti and Cisternas (2020) for the same properties in simplified monopoly settings.

[28] Note that, regardless of Bertrand or Cournot competition, the equilibrium quantities are increasing in the intercept of the own product (Y_i) and decreasing in the intercept of the others' product (Y_j) as long as the products are substitutes. In other words, the demand substitute property dominates, even with the strategic substitute or complement that arises depending on whether the competition is in prices or quantities.

the product substitution.

Even with such product substitution and covariance term, which is specific in the oligopoly setting, we can show that the positive α_i in the manipulation strategy increases the summation of the last three terms in Equation (21), resulting in the following proposition.

Proposition 7 (Consumer Surplus Comparison: With and Without Review Manipulation)**.** *If there exist manipulated reviews in the market and the representative consumer is rational, the ex-ante expected consumer surplus becomes higher than the one without manipulated reviews.*

The effect of the reputation manipulation on the profit is more subtle. While the existence of manipulation makes the rating more informative, the manipulation is costly for the seller. From numerical analysis reported on the bottom panels in Figure 4, we observe the following.

Remark 1. *Sellers face lower profits with reputation manipulation than those without.*

Even though the signaling role of review manipulation benefits the seller, the effect is relatively minor. Thus, the review manipulation is essentially wasteful for the firm, as in the previous research using career concern models.[29]

The effects of the substitution (competitiveness) parameter s on the surpluses are also analyzed numerically. Figure 4 reports the ex-ante variance of reputation ($Var(Y_i)$), ex-ante utility of the representative consumer ($E[U]$), and ex-ante profit per firm ($E[\pi_i]$). Those variables of interest are on the vertical axes, while the substitution parameter (s), review manipulation cost parameter (ϕ), and signal-to-noise ratio ($\sigma_\theta/\sigma_\epsilon$) are on the horizontal axes. Below, we focus on the blue lines, which illustrate the representative rational consumer.

As s increases, the ratings become more informative (the top-left panel) due to the increased α_i (on the top-left panel in Figure 3). Then the ex-ante consumer surplus increases (middle-left panel). Lastly, the bottom-left panel shows the ex-ante profit per firm, where a relatively higher product substitutability has two channels. First, on average, a higher product substitutability leads to severe competition in the pricing stage, resulting in a smaller in variable profit. Second, however, the amount of manipulated reviews decreases (the middle-left panel of Figure 3), and the firm can save on the manipulated-review writing cost. In a reasonable range of parameters, the first channel is dominant and profit is decreasing in s.[30]

[29]As long as the manipulation is possible, the consumer discounts the ratings, expecting the seller's manipulative behavior. Then, the seller has to manipulate the ratings to keep its reputation level.

[30]When the products become almost homogeneous, the price competition tends to predict negative quantity for a less attractive product. In the numerical analysis, we only report results where the probability of the negative quantity is sufficiently small (less than 10 percent).

In the center column in Figure 4, we report the equilibrium outcomes with respect to the review manipulation cost parameter (ϕ). This illustrates that, as ϕ increases, both the positive effect on the consumer surplus and the negative effect on the profit diminish. The top-center panel shows that the informativeness of the rating captured by the variance of reputation ($Y_i = E_c[\theta_i|R_i]$) decreases as ϕ_i goes up (and then, α_i decreases). Then, the *ex-ante* consumer surplus slightly decreases, while a degree of such an effect is almost negligible in the numerical analysis (see the middle-right panel). The bottom-center panel shows that the ex-ante expected profit increases when the review manipulation cost goes up. The manipulated reviews do not increase the expected price or quantity for any ϕ as shown in Proposition 5. The positive effect of the manipulation on the revenue via $Var(Y_i)$ is limited. The firm can commit to save the manipulation cost when the cost parameter becomes large (see the bottom-center panel of Figure 3), resulting in an increase in its expected ex-ante profit.

In the right column of Figure 4 reports the effect of signal-to-noise ratio. In the top-right panel, when the signal-to-noise ratio increases (i.e., when a rating conveys relatively more accurate information), as a natural consequence, the informativeness of the rating ($Var[Y_i]$) increases even in a benchmark without manipulation (black dash-dotted line). Furthermore, a positive α_i slightly shifts up the informativeness. Those effects result in a slightly increasing consumer surplus in the middle-right panel. By contrast, firms' profits are decreasing in the signal-to-noise ratio (bottom-right panel). While the firm tries to manipulate the reputation more as the signal-to-noise ratio increase (see the bottom-right panel of Figure 3), rational consumers curve ratings (thus, the expected revenue remains almost unchanged). Thus, the firm's profit decreases on average due to the increased manipulation cost.

5 Behavioral Consumers

In this section, we report the analyses of oligopoly models with naive consumers, who do not recognize the hidden review manipulation activities strategically conducted by firms. Given the existence of such naive consumers, firms adjust their review-manipulation strategies, and resulting market outcomes could be largely different from those with rational consumers.

5.1 Naive Consumers

We consider consumers who do not take the existence of manipulated reviews into account, while they understand that the reviews are noisy signals of underlying product qualities.[31]

[31] See also Eyster and Rabin (2005), Kartik et al. (2007), Murooka and Yamashita (2025), and Yasui (2020), who apply similar definitions of naive consumers. Upon considering behavioral consumers, for sim-

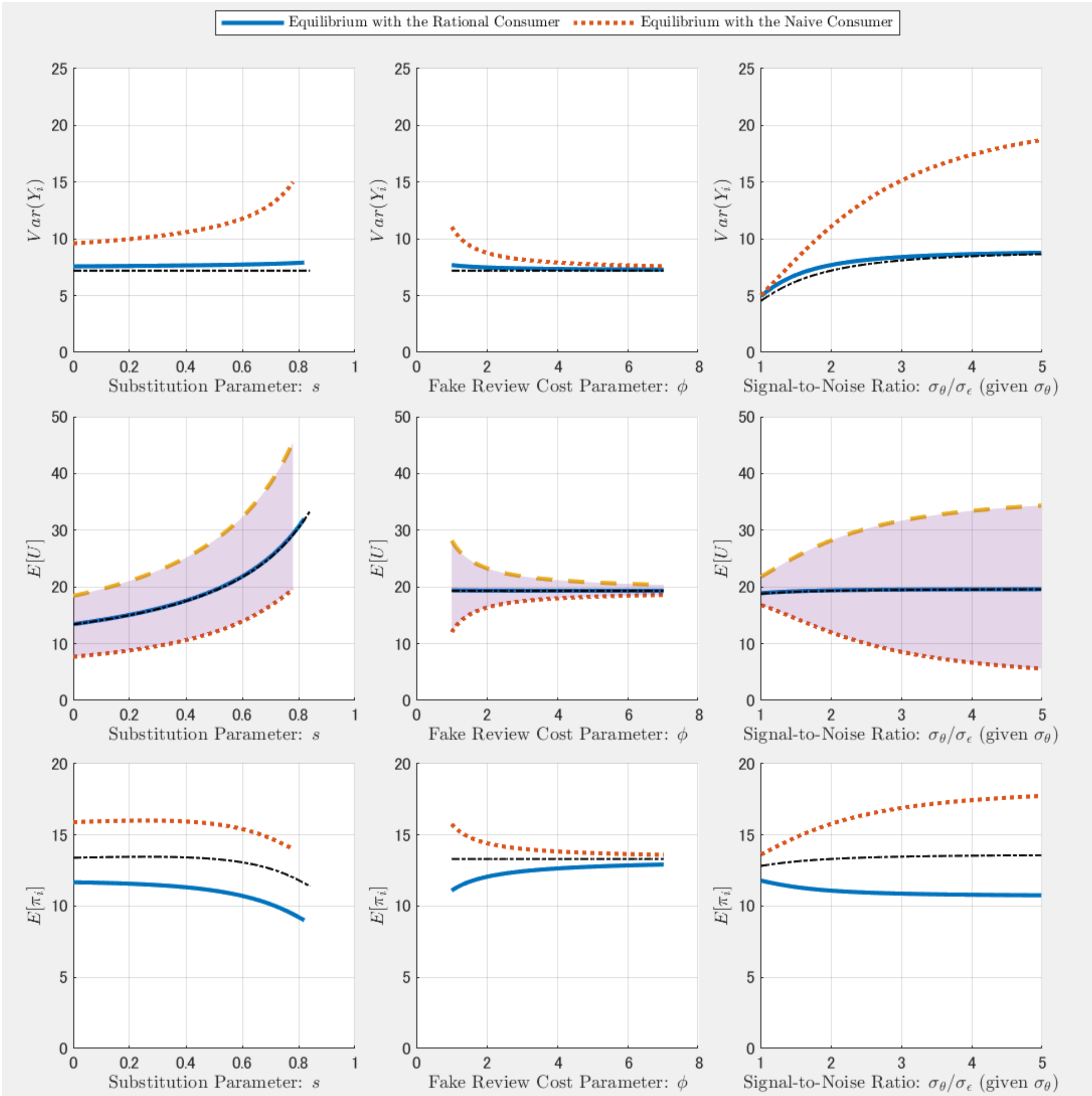

Figure 4: Comparative Statics - Equilibrium Welfare

Such a consumer's naive conjecture in products' qualities corresponds to a belief in θ_i, given $\alpha_i = 0$ and $\gamma_i = 0$. Even with such a belief in strategies, the naive consumer perceives a hidden type θ_i and rating R_i to be normally distributed. Therefore, we can still apply the projection (i.e., inversion) formula for the conditional expectation (of hidden product

plicity, the main part of this article focuses on the purely naive representative consumer, who faces limitations in curving an inflated rating. However, in the descriptions and proofs reported in the Appendix, to ensure comprehensiveness, we explore a partially naive representative consumer, who nests both rational and purely naive consumers as extreme cases. Specifically, a partially naive representative consumer's belief is represented as a convex combination of rational and purely naive representative consumers' beliefs.

quality) with the naive belief as

$$E_c^N[\theta_i|R_1,\,R_2] = \mu + \frac{Cov_c^N[\theta_i, R_i]}{Var_c^N[R_i]}\left(R_i - E_c^N[R_i]\right), \tag{22}$$

where superscripts N represent moments based on the naive consumer's conjecture on a seller's strategy ($\alpha_i = 0$ and $\gamma_i = 0$ in $F_i = \alpha_i\theta_i + \gamma_i$). Now, recall that the actual rating consists of $R_i = \theta_i + F_i + \epsilon_i$. However, with the naive belief in a seller's strategy ($\alpha_i = 0$ and $\gamma_i = 0$), the naive consumer does not curve an inflated rating in their expectation,

$$E_c^N[R_i] = E_c^N[\theta_i + F_i + \epsilon_i] = E[\theta_i] = \mu, \tag{23}$$

indicating that the naive consumer believes an observed rating is genuine. Similarly, the naive consumer views $Cov_c^N[\theta_i, R_i]$ and $Var_c^N[R_i]$ as coming from a joint normal distribution of θ_i and R_i with $\alpha_i = 0$ and $\gamma_i = 0$. Thus, we have a curving coefficient of the naive consumer,

$$\frac{Cov_c^N[\theta_i, R_i]}{Var_c^N[R_i]} = \frac{\sigma_{\theta,i}^2}{\sigma_{\theta,i}^2 + \sigma_{\epsilon,i}^2} = \frac{1}{1 + (\sigma_{\theta,i}/\sigma_{\epsilon,i})^{-2}} \equiv \underbrace{\lambda_i^N}_{\substack{\text{curving coefficient for firm } i \\ \text{among naive consumers}}}, \tag{24}$$

which contrasts with that of the rational consumer (see Equation (5)). Then, given Equations (23) and (24), the conditional expectation formed by the naive consumer (Equation (22)) is rewritten as

$$E_c^N[\theta_i|R_1,\,R_2] = \mu + \lambda_i^N(R_i - \mu) \equiv \underbrace{Y_i^N}_{\substack{\text{firm } i\text{'s reputation} \\ \text{among naive consumers}}}, \tag{25}$$

which we call *naive reputation* (or reputation among naive consumers). Such a naive reputation is biased by $\lambda_i^N E[F_i]$ on average (i.e., $E[E_c^N[\theta_i|R_1,\,R_2]] = \mu + \lambda_i^N E[F_i] \neq \mu = E[\theta_i]$). In addition, the marginal effect of rating on the reputation is greater with the naive consumer than with the rational consumer (i.e., $\partial Y_i^N/\partial R_i = \lambda_i^N > \lambda_i = \partial Y_i/\partial R_i$).

Given the observed ratings, at Stage 5, the naive consumer maximizes their *interim* expected utility with their naive reputations (Y_i^Ns) described in Equation (25), which is written as

$$E_c^N[U|\mathbf{R}, \mathbf{p}] = E_c^N[\theta_1|\mathbf{R}, \mathbf{p}]\,q_1 + E_c^N[\theta_2|\mathbf{R}, \mathbf{p}]\,q_2 - \frac{b_1}{2}q_1^2 - \frac{b_2}{2}q_2^2 - sq_1q_2 - p_1q_1 - p_2q_2. \tag{26}$$

Naive consumer's interim utility maximization results in the following first-order conditions:

$$\begin{aligned} p_1 &= Y_1^N - b_1 q_1 - s q_2 \\ p_2 &= Y_2^N - b_2 q_2 - s q_1. \end{aligned} \tag{27}$$

Thus, the only difference from the demand system with the rational consumer (Equation (8)) is the definition of naive reputations Y_i^Ns. Based on the above demand system, given the opponent's strategy, each firm chooses its optimal price (or quantity) in Stage 4. This results in the profit function quadratic in q_i^* as in Equation (9), but with a slightly different expression of q_i^*

$$q_i^* = L_i \left(Y_i^N - c_i\right) - M_i \left(Y_j^N - c_j\right), \tag{28}$$

where q_i^* is now a linear function of $(Y_i^N - c_i)$ and $(Y_j^N - c_j)$. Then, at Stage 2, as in the case with the rational consumer, a seller maximizes its conditional expected profit in Equation (11), where q_i^* is now characterized by Equation (28). Consequently, the same algebra up to Equation (15) holds, with a different expression of $E[q_i^*]$

$$E[q_i^*] = L_i \left(E[\theta_i] + \lambda_i E[F_i] - c_i\right) - M_i \left(E[\theta_j] + \lambda_j E[F_j] - c_j\right), \tag{29}$$

the first-order condition w.r.t. F_i characterizes the equilibrium strategy $F_i^*(\cdot)$. Next, α_i and λ_i are characterized by matching the relevant coefficient in Equation (15), combined with Equation (24), which now become (are) irrelevant to firm j's parameters. By contrast, $E[F_i]$ is characterized by taking expectation of Equation (15) and inserting Equation (29), we have

$$(\phi_i - 2\beta_i^{-1} L_i^2 \lambda_i^2) E[F_i] = 2\beta_i^{-1} L_i^2 \lambda_i \left((E[\theta_i] - c_i) - \frac{M_i}{L_i} (E[\theta_j] + \lambda_j E[F_j] - c_j) \right). \tag{30}$$

In contrast to the model with the rational consumer, the above equation is a decreasing function of $E[\theta_j] (j \neq i)$. Finally, γ_i is characterized by Equation (19). The characterization of the equilibrium manipulation strategy is now summarized in the following proposition.

Proposition 8 (Best response with the naive consumer)**.** *If the consumer is naive, then*

1. *α_i is positive and uniquely determined, and does not depend on the other firm's review manipulation strategy;*

2. *$E[F_i]$ is decreasing in $E[F_j]$ $(j \neq i)$*

The second property elucidates that, with the naive consumer, the amount of manipulated reviews shows a *strategic substitute* property. The intuition is as follows. When the opponent firm (firm j) generates a relatively large amount of manipulated reviews, naive consumers are credulously attracted by the opponent's product due to the inflated review rating of R_j. Then, the residual demand of firm i falls, resulting in a smaller marginal gain from a costly review manipulation effort, and firm i carries out less manipulation.

Regarding the existence of equilibrium, similar to the case with the rational consumers, in our analysis, we consider an equilibrium where firms' best responses intersect at once, described in the following proposition.

Proposition 9. *If the representative consumer is naive, the equilibrium exists and is unique.*

Note that, in contrast to the model with the rational consumer, we do not need a condition for the signal-to-noise ratio (which is$\sigma_\theta/\sigma_\epsilon > 1$). We can show this by slightly modifying the proof of Proposition 3. The equilibrium is unique because a curving coefficient λ_i^N among naive consumers is invariant to α_i (see Equation (24)), while in general α_i is increasing in λ_i (c.f., Figure 4). Given the unique α_i and λ_i for $i = 1, 2$, now $E[F_i]$ is determined as a solution of mutual best responses in Equation (30), which gives a unique solution due to the strategic substitute property.

The consumers' naivety also alters the equilibrium variables and properties. Notably, the effects of competitiveness on the firm's review manipulating behavior are largely different from those with the rational consumer (see red dotted lines in Figure 3), as articulated below.

Proposition 10. *If the representative consumer is naive, α_i and λ_i are greater than with the rational consumer.*

A sketch of the proof is as follows. The curving coefficient for the naive consumer λ_i^N is an upper-bound of the curving coefficient for the rational consumer, and thus, λ_i is greater with the naive consumer. Furthermore, because α_i is increasing in λ_i^N (or λ_i, depending on a naive or rational case), α_i is greater with the naive consumer. Intuitively, the naive consumer does not curve the inflated rating and reacts to the rating more sensitively than the rational consumer does.

As to the expected amount of the manipulative effort $E[F_i]$, there are two channels. Assuming the opponent's strategy is fixed, the high sensitivity to the ratings incentivizes firm i to manipulate its own rating. However, in equilibrium, the other firm faces the same incentive and would also do so, which could diminish firm i's profit, as well as reduce firm i's incentive to manipulate the rating. The following remark summarizes the properties of the ex-ante expected amount of manipulated reviews from the numerical analysis.

Remark 2. *If the consumer is naive, $E[F_i]$ is larger than one with the rational consumer.*

Thus, the incentive to attract the naive consumer dominates the strategic substitute effect.

5.2 Welfare Analysis with Naive Consumers

Finally, we analyze the effect of manipulated reviews on the naive representative consumer's surplus, notably by making the comparison with that of the rational consumer. On one hand, because α_i is still positive in equilibrium, manipulated reviews enhance the correlation between a firm's product rating R_i and the underlying hidden quality θ_i, which benefits the naive consumer. On the other hand, unlike the rational consumer, the naive consumer suffers from the bias in products' reputations (Y_i^Ns). Such a trade-off is summarized in the following expression of the *ex-ante* (before Stage 1) expected consumer surplus of the representative naive consumer, which is[32]

$$\begin{aligned} E[U] = \quad & \frac{b_1}{2} E\left[q_1^N\right]^2 + \frac{b_2}{2} E\left[q_2^N\right]^2 + s \cdot E\left[q_1^N\right] E\left[q_2^N\right] \\ & + \frac{b_1}{2} Var\left[q_1^N\right] + \frac{b_2}{2} Var\left[q_2^N\right] + s \cdot Cov\left[q_1^N, q_2^N\right] \\ & - \underbrace{E\left[\left(Y_1^N - Y_1^R\right) q_1^N + \left(Y_2^N - Y_2^R\right) q_2^N\right]}_{\text{damage caused by biases}}, \end{aligned} \tag{31}$$

where we use the notation of q_i^Ns for the naive consumer's equilibrium consumption levels, and Y_i^Rs for reputations formed by rational consumers (see Equations (4) and (7)), which serve as benchmarks. The first and second lines, which are essentially the same formula as those for the rational consumer (see Equation (21)), are interpreted as the consumer surplus *observed via the consumption.* As explained in Figure 4, these terms are even larger than those of the rational consumer. The naive consumer consumes larger amounts and reacts more strongly to ratings,and thus, from the viewpoint of revealed preference, they seemingly gain a higher consumer surplus. However, such a reaction is partly due to the biases caused by manipulated reviews. In the third line, the term of $Y_i^N - Y_i^R$ captures the bias, and $(Y_i^N - Y_i^R)q_i^N$ corresponds to the surplus damage caused by manipulated reviews. The naive consumer places higher (and biased) reputations on the products' qualities than the rational consumer does. The harm to the naive consumer is even greater if the consumer purchases a larger volume of the products with biased reputations. In the numerical exercises reported in Figure 4, such biases (in the third line of Equation (31)) dominate the positive effect (in the first and second lines) of manipulated reviews on the naive consumer. We also implement numerical comparative statics with respect to the competitiveness (product-substitution) parameter s and review-manipulation cost parameter ϕ_i, resulting in the following remark.

Remark 3. *If the consumer is naive,*

[32]See the Appendix for the derivation.

1. *the consumer surplus is lower with manipulated reviews than without them;*

2. *the consumer surplus increases in ϕ_i.*

The first point shows that the damage caused by biased reputations dominates the positive effect of manipulated reviews. Thus, the naive consumer, on average, faces a lower consumer surplus compared to that of the rational consumer. The second point is a natural consequence of the increase in the marginal cost of review manipulation. As the cost for writing manipulated reviews increases, a firm writes fewer manipulated reviews, resulting in less damage to the naive consumer surplus.

We now report comparative statics on the equilibrium coefficients with naive consumers, in comparison with rational consumers. In Figure 3, dotted red lines depict the properties of equilibrium coefficients with naive consumers, and solid blue lines depict those with rational consumers, as described in the previous section.

First, in the left column of Figure 3, we report the effects of the product substitution (competitiveness) parameter, s, which a market regulator may have leverage over.[33] In the Left-Middle Panel, as articulated in Equations (22) and (24), the curving coefficient (λ_i^N) is constant at one without the manipulation, as the naive consumers cannot take the manipulation into account. In the Left-Top Panel, however, a firm in equilibrium reacts more to its type through its linear strategy (i.e., $F_i = \alpha_i\theta_i + \gamma_i$). This firm-side behavior is because a slightly higher rating can attract relatively more demand from naive consumers when products become more substitutable, notably compared to the case with rational consumers. In the Left-Bottom Panel, the ex-ante expected amount of manipulated reviews with naive consumers (dotted red line) exceeds that of those generated with rational consumers (solid blue line). This predominance is because the naive consumer capturing role of the manipulated reviews (as depicted in the Left Top Panel) far exceeds the signaling role of the manipulated reviews, as manifested by the widening gap between the dotted red line (with naive consumers) and the solid blue line (with rational consumers) with respect to the product substitution parameter, s.

Second, in the center column of Figure 3, we report the equilibrium coefficients with respect to a varying review-manipulation cost (ϕ_i).[34] In the Center-Middle Panel, for the same reason as in the Left-Middle Panel, the curving coefficient is constant. As a matter of course, the linear strategy coefficient (α_i) and the expected level of manipulation ($E[F_i]$) with naive consumers diminish when the review manipulation cost moves up (Center-Top

[33]For example, a regulator may instruct online platforms to make product comparison more salient by redesigning the platform website display.

[34]For instance, the authority could regulate online platforms to hire more programmers to detect and remove manipulated reviews, which increases the cost of the review-manipulation.

and Center-Bottom Panels).

Third, in the right column of Figure 3, we illustrate the comparative statics when the review noise ($\sigma_{\epsilon,i}$) in a rating becomes smaller. Given the variance of quality ($\sigma^2_{\theta,i}$) type is fixed, a regulatory intervention to impose a smaller review noise is equivalent to making the signal-to-noise ratio ($\sigma_{\theta,i}/\sigma_{\epsilon,i}$) larger. In the Right-Middle Panel, when the naive consumers are exposed to smaller review noise (i.e., larger $\sigma_{\theta,i}/\sigma_{\epsilon,i}$), they believe that the observed rating (R_i) transmits relatively more accurate information about the true product quality (i.e., higher λ_i^N). From the viewpoint of sellers, given such a curving process by naive consumers, the marginal gain of manipulative behavior increases as the rating becomes more accurate, and then, both the level of manipulative behavior ($E[F_i]$) and its signaling effect (α_i) are enhanced.

Next, the welfare consequences of manipulation reviews are vital for market regulation. In Figure 4, reports the informativeness and welfare-related numerics as red dotted lines with varying model primitive parameters. Moreover, we report the naive consumer surplus seemingly revealed via observed prices and quantities (i.e., the first and second lines of a representative naive consumer's ex-ante surplus in Equation (31)) as the dashed yellow line, and define the shaded area as a representative naive consumer's utility loss caused by their behavioral bias (see the third line in Equation (31)).

First, in the left column of Figure 4, we report the comparative statics with respect to the substitution parameter, s. In the Left Top Panel, when the product becomes more substitutable with a larger s, a firm reacts more to its type (see the Left Top Panel of Figure 3) by choosing a higher α_i (in linear strategy $F_i = \alpha_i\theta_i + \gamma_i$), and the variance of reputation $Var[Y_i]$ (i.e., informativeness of rating) increases. However, as depicted in the Left Middle Panel, naive consumers overreact to the observed rating, causing overconsumption (see the first and second lines in Equation (31)). This overconsumption results in lower utility compared to that of rational consumers (the solid blue line), although their utility improves as the substitution parameter becomes larger (i.e., as a rating becomes more informative). On the other side of the market, as depicted in the Left Bottom Panel, when products become more substitutable, the ex-ante expected profit of a firm with naive consumers becomes smaller due to the intensified competition.

Second, in the center column of Figure 4, we consider the increase in review manipulation cost (ϕ_i). In the Center Top Panel, when the review manipulation cost goes up, the firm becomes less reactive to its type as the marginal cost of review manipulation increases. As a result, naive consumers are exposed to fewer manipulated reviews (see the Center Bottom Panel in Figure 3), leading to a smaller surplus loss from reduced behavioral bias, as reported in the Center Middle Panel of Figure 4. In consequence, in the Center Bottom

Panel, the firm's profit diminishes when the review manipulation cost goes up, as it can generate smaller profit from the exploitation of naive consumers with fewer manipulated reviews, and this smaller exploitation of naive consumers effect exceeds the cost saving effect through generating fewer fake reviews.

Third, in the right column of 4, we analyze the effect of diminished review noise ($\sigma_{\epsilon,i}$), which is equivalent to the effect of increased noise-ratio (i.e., an increase in $\sigma_{\theta,i}/\sigma_{\epsilon,i}$, given the $\sigma_{\theta,i}$ fixed). When the review noise diminishes (and noise-ratio increases), the firm becomes more reactive to its type, represented by a larger α_i (see the Center Top Panel in Figure 3). As a result, in the Center Top Panel in Figure 4, the variance of reputation $Var[Y_i]$ (informativeness of the rating) goes up when the review noise shrinks. However, the naive consumers largely overreact to the observed rating, which is now more informative, resulting in a larger utility loss, as reported in the Right Middle Panel. On the other side of the market, in the Right Bottom Panel, the firm earns a higher profit as the informativeness of ratings increases, notably due to naive consumers' overreaction to ratings and their overconsumption.

Lastly, regarding market regulatory policies for improving naive consumers' surplus, the middle row of Figure 4 reports the drastic difference in the consequences of regulatory intervention, notably by clarifying several channels for either improving or inadvertently damaging naive consumers' surplus. The most notable contrast is that, while the interventions to increase product substitution (s) and review manipulation cost (ϕ_i) improve naive consumers' surplus, the reduction of review noise ($\sigma_{\epsilon,i}$) damages it. Moreover, even when surplus improvement occurs, the channels of improvement differ considerably, particularly in terms of the amount of manipulated reviews and how informative an observed rating (R_is) is. Specifically, greater product substitutability (s) increases the amount of manipulated reviews and positively influences how much information a rating provides (i.e., a larger $Var(Y_i)$) for naive consumers. However, the increase in review manipulation cost (ϕ_i) decreases the amount of manipulated reviews and reduces how much information a rating provides (i.e., a smaller $Var(Y_i)$), again for naive consumers. Lastly, and most drastically, although the informativeness of ratings increases, the regulatory intervention to make review noise ($\sigma_{\epsilon,i}$) smaller damages naive consumer surplus. This damage occurs because naive consumers ingenuously follow more informative (but more manipulated) ratings, resulting in overconsumption (dashed yellow line in Right-Middle Panel in Figure 4) with a larger bias (shaded area in Right-Middle Panel).

6 Extensions

In this section, we report the three extensions of our baseline model with rational consumers introduced in Section 3. First, we discuss a competition between two *ex-ante asymmetric* firms, where firms draw their product qualities from asymmetric distributions.We report a condition where ex-ante lower-quality firms write more manipulated reviews than ex-ante higher-quality firm does. The second extension analyzes the *Cournot quantity competition* among two ex-ante symmetric firms. The third extension regards an oligopoly competition among n *ex-ante (semi-)symmetric firms*. This extension allows us to report some asymptotic (when the number of firms becomes $n \to \infty$) results. In this section, we report the extended models with the rational consumers for simplicity, while the proofs in the Appendix are in a general form including both the rational and naive consumer cases.

6.1 (*Ex-Ante*) Asymmetric Quality/Signal

In this section, we discuss *ex-ante* asymmetric distributions of qualities, as well as rating shocks, for sellers. In a two-firm price competition with rational consumers, each seller draws its quality type (θ_i) independently from $\mathcal{N}(\mu_i, \sigma^2_{\theta,i})$, as well as drawing the noise of the rating (ϵ_i) independently from $\mathcal{N}(0, \sigma^2_{\epsilon,i})$. We can interpret that one firm has established its reputation better than the other. For an online sales example, if firm $i \in \{1, 2\}$ has established a better reputation than firm $j \neq i$, outside this online market, the asymmetry is captured by $\mu_i > \mu_j$. If firm i collects more reviews on the platform than firm j, that is captured by $\sigma_{\theta,i}/\sigma_{\epsilon,i} > \sigma_{\theta,j}/\sigma_{\epsilon,j}$. Importantly, the construction of ratings, payoff functions, and the definition of equilibrium do not change from the main model in Sections 3 and 4.

Even under this ex-ante asymmetric firms environment, θ_i is independent from R_j, as θ_i is independent from θ_j and ϵ_j. Thus, the seller i's reputation by the rational consumer (Y_i) is rewritten as

$$Y_i \equiv E_c[\theta_i|\mathbf{R}, \mathbf{p}] = E[\theta_i] + \frac{(1+\alpha_i)\sigma^2_{\theta,i}}{(1+\alpha_i)^2\sigma^2_{\theta,i} + \sigma^2_{\epsilon,i}}(R_i - \{E[\theta_i] + E_c[F_i]\}).$$

Now, by redefining $\lambda_i \equiv \frac{(1+\alpha_i)\sigma^2_{\theta,i}}{(1+\alpha_i)^2\sigma^2_{\theta,i}+\sigma^2_{\epsilon,i}}$, all the equilibrium characterizations in the main part hold. This leads to the straightforward extension of Proposition 1 to ex-ante asymmetric firms.

Proposition 11. *Suppose the consumers are rational. Then, at equilibrium,*

1. *α_i and λ_i depend neither on μ_j nor $\sigma_{\theta,j}/\sigma_{\epsilon,j}$ of the opponent firm $j \neq i$, and*

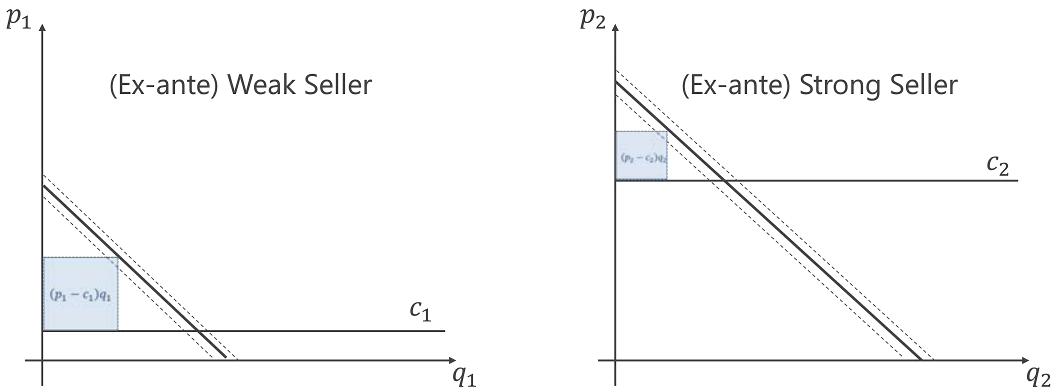

Figure 5: Asymmetric Firms: Given the Opponent's Quantity and Reputation Fixed

2. $E[F_i]$ is decreasing in μ_j, but it does not depend on $\sigma_{\theta,j}/\sigma_{\epsilon,j}$ for $j \neq i$.

As in the main part (Section 3), the first property comes from the fact that, in expectation, rational consumers can correctly curve the opponent's review-inflating manipulation. Thus, the opponents' parameters that affect $F_j^*(\cdot)$ do not affect seller i's sales not firm i's incentive to manipulate its reputation.

Regarding the second point, nevertheless, if the opponent has a high prior reputation, the seller i's expected revenue and the marginal revenue of manipulated reviews become small. Thus, seller i does not manipulate reviews much.

Regarding the anecdotal observation that a low-quality firm is more likely to write manipulated reviews, which looks to contradict the positive α_i at equilibrium, our asymmetric model provides an explanation.[35] Suppose $\mu_1 < \mu_2$ and $\sigma_{\theta,1}/\sigma_{\epsilon,1} = \sigma_{\theta,2}/\sigma_{\epsilon,2}$. That is, firm 1 has a better prior than firm 2. We label firm 1 as *(ex-ante) weak* and firm 2 as *(ex-ante) strong*.

Suppose also that the marginal review-manipulation costs are the same across two firms ($\phi_1 = \phi_2$). Given the opponent's quantity, Figure 5 depicts a situation in which a weak firm writes more manipulated reviews. In this setting, while the strong firm (firm 2) has a higher expected product quality, it also exhibits a higher marginal production cost c_2, resulting in a relatively smaller expected variable profit, depicted as a square area in the right diagram. On the other hand, the weak firm (firm 1) has a relatively low expected quality, yet its marginal production cost c_1 is also low, resulting in a relatively larger expected variable profit, as illustrated on the in Figure 5. As discussed in Section 4, the marginal gain from writing an additional manipulated review depends on the size of the expected variable profit (refer

[35] Note that, with the observational data of a market where the ratings play key roles, it is a challenging task to test whether a high-quality or low-quality seller manipulates reviews more. A limited number of empirical studies, such as He et al. (2022a), indicate that lower-quality firms write more manipulated reviews.

to the square function part in Equation (11). Thus, the ex-ante weak firm can have more incentive to manipulate reviews in this asymmetric duopoly situation, resulting in generating a relatively larger amount of manipulated reviews, given the same marginal manipulated review writing cost.

Alternatively, even when the production cost c_i's are the same among two firms, the review manipulation cost is relatively higher for the strong firm. In a case where reputation manipulation is revealed to the public, it might damage the strong firm's established reputation, resulting in less manipulative behavior by the strong firm.

6.2 Quantity Competition

In our main model explored in Section 4, we focused on the Bertrand price competition in a two-firm differentiated-product market. We now straightforwardly extend our model to a two-firm Cournot competition, by minimally modifying the profit function at the market competition stage (at Stage 4, see Equation (9)) with the Cournot version of the coefficients as explained in the Appendix C. Then, the equilibrium characterization and the comparative statics with respect to ϕ_i and $\sigma_{\theta,i}/\sigma_{\epsilon,i}$ are robust to the Cournot specification.

6.3 n-Firms Oligopoly

In this section, we extend our models to n-firm oligopoly competition, in which firms engage in price (or quantity) competition. We use the firm index of $i \in \{1, \cdots, n\}$, and each firm draws its hidden product quality type θ_i from an i.i.d. distribution of $\mathcal{N}(\mu, \sigma_\theta^2)$.[36] For the sake of tractability, we focus on a linear demand system with symmetric slope and substitution parameters (i.e., common b and s across all firms). On the other hand, we allow the demand intercepts and marginal production costs to be asymmetric (i.e., heterogeneous marginal production costs, c_is). Timing and other assumptions (such as the linear rating of $R_i = \theta_i + F_i + \epsilon_i$) remain the same as those in the main model presented in Section 3. Then, similar to the two-firm case, the profit function for each firm (at Stage 2) is still reduced to a quadratic function of its own reputation ($Y_i = E_c[\theta_i|R_i]$) and other firms' reputations ($Y_j = E_c[\theta_j|R_j]$s, where $i \neq j$).[37] As before, we can solve the equilibrium by focusing on linear strategies, $F_i = \alpha_i\theta_i + \gamma_i$, for each firm.[38]

[36] The i.i.d. assumption can be easily extended to asymmetric independent distribution as in Section 6.1.

[37] If we allow asymmetric demand slope and substitution parameters across firms, the profit functions can still be expressed in a quadratic form (of firm i's reputation or its review-manipulating action) with a heterogeneous demand slope matrix, although the analysis becomes more intricate. See Choné and Linnemer (2020) for a summary of price or quantity competition with a general linear demand function.

[38] As in the main part with two firms, firms' strategy space in review manipulation is not restricted to linear strategies. We guess a linear strategy equilibrium and such an equilibrium is verified to exist.

Under this n-firm oligopoly setting, we assume a representative consumer with the following *ex-post* utility function:

$$U = \theta' \mathbf{q} + \frac{1}{2} \mathbf{q}' \mathbf{B} \mathbf{q} - \mathbf{p}' \mathbf{q}, \tag{32}$$

where $\theta = (\theta_1, ..., \theta_n)'$, $\mathbf{q} = (q_1, ..., q_n)'$, $\mathbf{p} = (p_1, ..., p_n)'$, and $\mathbf{B}$ is n-by-n symmetric matrix with b's in its diagonal elements and s's in its off-diagonal elements. Then, at the beginning of Stage 5, given the ratings $\mathbf{R} = (R_1, ..., R_n)'$, the consumer faces the *interim* expected utility as in Spence (1976):

$$E_c[U|\mathbf{R}, \mathbf{P}] = \mathbf{Y}' \mathbf{q} + \frac{1}{2} \mathbf{q}' \mathbf{B} \mathbf{q} - \mathbf{p}' \mathbf{q}, \tag{33}$$

where we use the notation of $\mathbf{Y} = (Y_1, ..., Y_n)' = (E_c[\theta_1|R_1], ..., E_c[\theta_n|R_n])'$. The inverse demand function for product i is derived by the first-order condition of the representative consumer with respect to q_i:

$$p_i = Y_i - bq_i - s \sum_{j \neq i} q_j. \tag{34}$$

Given the above demand function, the price (or quantity) competition results in a profit function quadratic in its own equilibrium quantity q_i^*, which is linear in own and others' reputation:

$$q_i^* = L_i (Y_i - c_i) - M_i \sum_{j \neq i} (Y_j - c_j). \tag{35}$$

where L_i and M_i are positive constants, although their exact values vary across different model primitive parameter settings and whether the competition is in prices or quantities.[39] Then, by replacing $M_1(Y_2 - c_2)$ in Equation (9) in the main part (in Section 4) with $M_i \sum_{j \neq i}(Y_j - c_j)$, we can generalize all the results so far to the n-firms oligopoly setting. Furthermore, we can analyze how the review manipulation strategy and the surplus change as the number of firms increases.

Note that if we keep the parameters in Equation (33) constant and increase the number of sellers n, given the prices fixed, the total consumption and the consumer surplus mechanically increase, because of the increased variety of the products. We eliminate such an effect and focus on strategic effect of the increased number of sellers by replacing b with $(n-(n-1)s)/\beta$ and s with s/β where β is an diagonal element of $\mathbf{B}^{-1}$ (i.e., the own price effect on the

[39] See the Appendix C for the derivation.

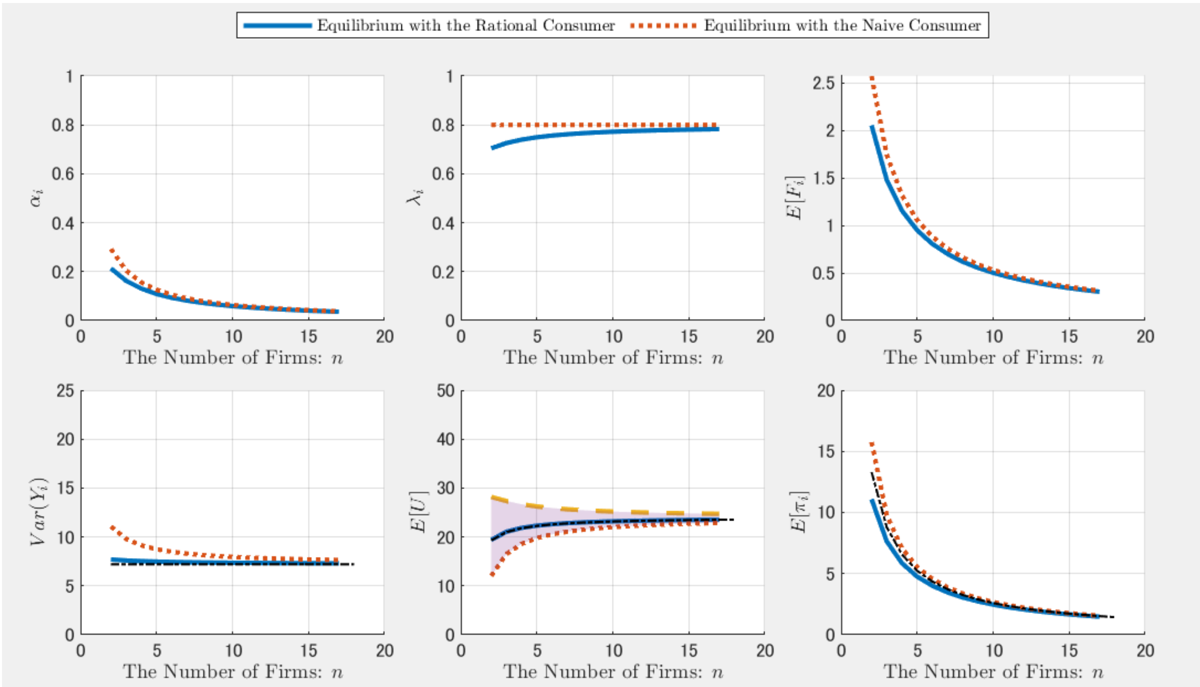

Figure 6: n-Firm Bertrand Competition with Rational and Naive Consumers

demand).[40]

In Figure 6, we illustrate the quantified equilibrium outcomes with varying numbers of competing firms in an oligopoly market. With this figure, we question whether the traditional market-competition policy of encouraging more competitors would improve consumer surplus, notably from the perspective of naive consumers. Before proceeding, as a strong caveat, even without manipulated reviews, consumer surplus must increase simply because of the intensified strategic interaction in a Bertrand differentiated-product oligopoly market. Thus, from a consumer protection perspective, the core question is whether an increase in n would bring the naive consumers' surplus closer to that of rational consumers (which is the the solid blue line in Figure 6), or close to a consumer surplus level in a market without manipulated reviews (which is the dotted black line in Figure 6). We quantitatively address this policy question, though we need to pay attention to the welfare-improvement channel related to the informativeness of observed ratings.

In the top-right panel of Figure 6, when the number of firms increases, the amount of the manipulated review generated decreases, as summarized by the following remark, which is directly relevant to competition policy.

Remark 4. *With both rational and naive consumers, when there are n firms in a Bertrand*

[40]See Shubik and Levitan (1980) for a summary of their work related to this, and see also Choné and Linnemer (2020) for a summary of the literature.

competition, ex-ante expected amount of manipulated reviews $E[F_i]$ shrinks as n goes up.

This remark implies that the traditional market competition policy, encouraging more competitors, deters the manipulated reviews. This decrease is because, given tighter product-market competition, the marginal gain function shrinks with n, while the marginal cost of manipulation ($\phi_i F_i$) remains the same due to its invariance to n. Regarding the marginal gain, as n increases, while an additional manipulated review can steal demand from a larger number of competitors, each oligopoly firm's expected markup (i.e., the markup after generating an additional manipulated review) shrinks due to tighter competition. As the number of competitors increases, the latter shrinks faster than the former, resulting in a declining marginal gain from review manipulation.

In the bottom-center panel of Figure 6, we observe a gradually increasing naive consumers' surplus (dotted red line) as consumers are less exposed to manipulated reviews for each firm. Eventually, when the number of competing firms becomes large enough, the ex-ante expected naive consumer surplus (dotted red line) becomes quantitatively similar to that of rational consumers (solid blue line), as well as the level without review manipulation (dotted black line). On the other side of the market, in the bottom-right panel, the ex-ante expected profit per firm shrinks faster with naive consumers, as the firm generates fewer manipulated reviews and thus cannot exploit their naivety, as n increases. However, regarding the channel of consumer surplus improvement, when n goes up, as each firm generates fewer manipulated reviews, the variance of reputation $Var[Y_i]$ (informativeness of a rating) declines, indicating the signaling role of a review rating dwarfs in the oligopoly market, as illustrated in the bottom-left panel.

7 Conclusion

We analyze reputation manipulation, such as fake reviews on online transaction platforms, in an oligopoly model with hidden information on product qualities, which have not been reported in the literature. The presence of review manipulation could alter the oligopoly asymmetric information market competitions and their outcomes. Resulting market consequences are of the interest among researchers, as well as informative to market regulators and competition authority policymakers. In our models, each firm engages in a costly review-manipulating action, inflates its review rating, and attempts to make its product look more attractive than others. On the other side of the market, given available rating information, consumers rationally or naively infer the quality of the product offered by each firm. Specifically, with an inflated rating, a firm could raise consumers' expected willingness to pay for its product, potentially upwardly shifting its demand function. Given such an inflated demand system, firms then engage in a standard static oligopoly market competition.

Analytically, we propose a linear rating specification with an idiosyncratic review noise (see Holmström (1999)): A product rating consists of the linear combination of (1) a true product-quality type, (2) manipulated reviews, and (3) an idiosyncratic review shock. (1) and (3) are from independent normal distributions, while (2) is the subject of strategic manipulation by a firm. Then, given a linear demand system, and by focusing on a linear strategy in private information of product quality type, we report that linear equilibrium review manipulation strategies exist in Bertrand (and Cournot) competitions. Moreover, we apply the model to the market with both rational and naive consumers. Notably, regardless of price or quantity competition, and regardless of the consumers' rationality, there exists a positive monotone strategy for each firm: the equilibrium amount of manipulated reviews generated by an oligopoly firm increases with respect to a firm's hidden quality type.

With rational consumers, we report our benchmark result that manipulated reviews partially alter market outcomes: expected prices and quantities remain unchanged with review manipulation, while the second moments of prices and quantities and expected surpluses are altered. With rational consumers, we report the following four findings.

First, we report that a firm's profit-maximizing review manipulation strategy does not depend on other firms' review manipulation strategies. In equilibrium, after rational consumers observe a firm's rating, they rationally conjecture its product quality type, forming their willingness to pay for this firm's product. The rational consumers correctly curve the inflated review ratings. Given this conjecturing process, each firm chooses its manipulated review writing action based on the ex-ante expected type(s) of opponent firm(s) because the opponent's manipulated reviews affect only via consumers' expectation on the opponent's quality, which is correctly curved by rational consumers. Accordingly, in equilibrium, a firm does not need to condition its review manipulation strategy on the other firm's (other firms') manipulated reviews. In addition, manipulated reviews have a signaling property, indicating that a high-quality firm engages in more review manipulation, because high-quality leads to large sales even without its manipulated reviews, which then implies a large marginal impact of manipulated reviews.

Second, stemming from the signaling role of manipulated reviews with rational consumers described above, we report that expected market quantities and prices remain the same with and without manipulated reviews, regardless of the Bertrand or Cournot competition. Third, counter-intuitively, manipulated reviews improve rational consumers' surplus, as rational consumers could recognize observed ratings as signals and could infer the process for inflating ratings with costly review manipulation activities. In other words, ratings inflated by manipulated reviews could better signal product quality than ratings based only on authentic reviews. This surplus improvement is related to the consumer surplus (expected utility) with variance terms. The existence of manipulation results in a higher variance of review ratings, which is more informative to rational consumers for conjecturing product quality, compared to a no-manipulated-review environment. Fourth, with rational consumers, we report some competition policy implications, such as increased review manipulation costs.

Next, given the benchmark results with rational consumers, we extend our model to (partially) naive consumers, who cannot fully comprehend review-manipulating actions done by oligopoly firms. Note that partially naive consumers nest both rational and naive consumers as extreme cases. With (partially) naive consumers, the existence of manipulated reviews results in large distortions in oligopoly market outcomes. Specifically, we report a following takeaway.

With (partially) naive consumers, a strategic substitution property emerges in firms' review-manipulating actions, which is a contrast to the market outcome with rational consumers. When rival firms write a large amount of manipulated reviews, (partially) naive consumers believe that inflated reviews are genuine and are attracted by those rival firms' products. As a result, the firm in question faces a relatively diminished residual demand function due to the product substitutability. Then, the marginal return from a review manipulation action for this firm is now relatively smaller (than the one with rational consumers), resulting in a relatively smaller amount of review manipulation effort. Similar logic applies to the vice-versa case, consisting of strategic review-manipulation substitutability.

To illustrate these differences in oligopoly market outcomes with rational or (partially) naive consumers, we report numerical comparisons under duopoly competition. Specifically, under the Cournot and Bertrand competition, respectively, we quantify the ex-ante measurements of welfare: consumer surpluses (for representative rational or pure naive consumers) and profits. In addition, we calculate the amount of manipulated reviews generated. Our numerical comparison confirms a decreased consumer surplus among the pure naive consumers caused by manipulated reviews, compared to the rational consumers. We also confirm a large increase in profits of manipulation-review-writing firms when they sell their products to pure naive consumers. These quantified numbers have not been reported in previous oligopoly market studies and could be informative to the market regulating authorities.

Lastly, it is worth mentioning that the equilibrium oligopoly market outcomes crucially depend on consumers' rationality in conjecturing the costly review manipulation process. Given this, we would like to report that the scope of consumer rationality regarding the inflated ratings formation is the subject of experimental and empirical investigations (for example, see Akesson et al. (2023)), and we leave such applied topics for future studies, which are currently and actively being researched.

References

Akesson, Jesper, Robert W Hahn, Robert D Metcalfe, and Manuel Monti-Nussbaum, "The Impact of Fake Reviews on Demand and Welfare," Technical Report, National Bureau of Economic Research 2023.

Aköz, Kemal Kıvanç, Cemal Eren Arbatli, and Levent Celik, "Manipulation through biased product reviews," *The Journal of Industrial Economics*, 2020, *68* (4), 591–639.

Armstrong, Mark and John Vickers, "Which demand systems can be generated by discrete choice?," *Journal of Economic Theory*, 2015, *158*, 293–307.

Ball, Ian, "Scoring strategic agents," *American Economic Journal: Microeconomics*, 2025, *17* (1), 97–129.

_ , "Scoring Strategic Agents," *American Economic Journal: Microeconomics*, Forthcoming.

Bar-Isaac, Heski and Steven Tadelis, "Seller reputation," *Foundations and Trends® in Microeconomics*, 2008, *4* (4), 273–351.

Bergemann, Dirk and Alessandro Bonatti, "Markets for information: An introduction," *Annual Review of Economics*, 2019, *11* (1), 85–107.

Bonatti, Alessandro and Gonzalo Cisternas, "Consumer scores and price discrimination," *Review of Economic Studies*, 2020, *87* (2), 750–791.

Cambanis, Stamatis, Steel Huang, and Gordon Simons, "On the theory of elliptically contoured distributions," *Journal of Multivariate Analysis*, 1981, *11* (3), 368–385.

Campbell, Arthur, Dina Mayzlin, and Jiwoong Shin, "Managing buzz," *RAND Journal of Economics*, 2017, *48* (1), 203–229.

Chevalier, Judith A. and Dina Mayzlin, "The effect of word of mouth on sales: Online book reviews," *Journal of Marketing Research*, 2006, *43* (3), 345–354.

Chevalier, Judith A, Yaniv Dover, and Dina Mayzlin, "Channels of impact: User reviews when quality is dynamic and managers respond," *Marketing Science*, 2018, *37* (5), 688–709.

Choné, Philippe and Laurent Linnemer, "Linear demand systems for differentiated goods: Overview and user's guide," *International Journal of Industrial Organization*, December 2020, *73*, 102663.

DeGroot, Morris H, *Optimal statistical decisions*, John Wiley & Sons, 2005.

Dellarocas, Chrysanthos, "Strategic Manipulation of Internet Opinion Forums: Implications for Consumers and Firms," *Management Science*, 2006, *52* (10), 1577–1593.

Dixit, Avinash, "A model of duopoly suggesting a theory of entry barriers," *Bell Journal of Economics*, 1979, pp. 20–32.

Eyster, Erik and Matthew Rabin, "Cursed equilibrium," *Econometrica*, 2005, *73* (5), 1623–1672.

Gandhi, Ashvin and Brett Hollenbeck, "Misinformation and Mistrust: The Equilibrium Effects of Fake Reviews on Amazon.com," Technical Report, UCLA Anderson School 2023.

Glazer, Jacob, Helios Herrera, and Motty Perry, "Fake reviews," *The Economic Journal*, 2021, *131* (636), 1772–1787.

Gómez, Eusebio, Miguel A Gómez-Villegas, and J Miguel Marín, "A survey on continuous elliptical vector distributions," *Revista matemática complutense*, 2003, *16* (1), 345–361.

Grunewald, Andreas and Matthias Kräkel, "Advertising as signal jamming," *International Journal of Industrial Organization*, 2017, *55*, 91–113. Publisher: Elsevier B.V.

He, Sherry, Brett Hollenbeck, and Davide Proserpio, "The Market for Fake Reviews," *Marketing Science*, 2022, *41* (5), 896–921.

_ , _ , Gijs Overgoor, Davide Proserpio, and Ali Tosyali, "Detecting fake-review buyers using network structure: Direct evidence from Amazon," *Proceedings of the National Academy of Sciences*, November 2022, *119.*

Hollenbeck, Brett, "Online Reputation Mechanisms and the Decreasing Value of Chain Affiliation," *Journal of Marketing Research*, October 2018, *55* (5), 636–654. Publisher: SAGE Publications Ltd.

Holmström, Bengt, "Managerial Incentive Problems: A Dynamic Perspective," *Review of Economic Studies*, 1999, *66* (1), 169–182.

Hörner, Johannes and Nicolas Lambert, "Motivational Ratings," *Review of Economic Studies*, 2021, *88* (4), 1892–1935.

Hurkens, Sjaak, "Bayesian Nash equilibrium in "linear" Cournot models with private information about costs," *International Journal of Economic Theory*, 2014, *10* (2), 203–217.

Kartik, Navin, Marco Ottaviani, and Francesco Squintani, "Credulity, lies, and costly talk," *Journal of Economic theory*, 2007, *134* (1), 93–116.

Lizzeri, Alessandro, "Information revelation and certification intermediaries," *RAND Journal of Economics*, 1999, pp. 214–231.

Mayzlin, Dina, "Promotional Chat on the Internet," *Marketing Science*, 2006, *25* (2), 155–163.

_ **, Yaniv Dover, and Judith Chevalier**, "Promotional Reviews: An Empirical Investigation of Online Review Manipulation," *American Economic Review*, 2014, *104* (8), 2421–2455.

Milgrom, Paul and John Roberts, "Price and advertising signals of product quality," *Journal of Political Economy*, 1986, *94* (4), 796–821.

Murooka, Takeshi and Takuro Yamashita, "Optimal trade mechanisms with adverse selection and inferential naivety," *American Economic Journal: Microeconomics*, 2025, *17* (4), 33–67.

Nelson, Phillip, "Information and consumer behavior," *Journal of political economy*, 1970, *78* (2), 311–329.

_ , "Advertising as information," *Journal of political economy*, 1974, *82* (4), 729–754.

Reimers, Imke and Joel Waldfogel, "Digitization and pre-purchase information: the causal and welfare impacts of reviews and crowd ratings," *American Economic Review*, 2021, *111* (6), 1944–1971.

Sahni, Navdeep S and Harikesh S Nair, "Does advertising serve as a signal? Evidence from a field experiment in mobile search," *The Review of Economic Studies*, 2020, *87* (3), 1529–1564.

Schmalensee, Richard, "A model of advertising and product quality," *Journal of political economy*, 1978, *86* (3), 485–503.

Shubik, Martin and Richard Levitan, "Market Structure and Behavior," in "Market Structure and Behavior," Harvard University Press, 1980.

Spence, Michael, "Product Differentiation and Welfare," *The American Economic Review*, 1976, *66* (2), 407–414.

Yasui, Yuta, "Controlling fake reviews," *Available at SSRN 3693468*, 2020.

Yoshimoto, Hisayuki and Andriy Zapechelnyuk, "Dynamic incentives behind manipulated online reviews," *Working Paper*, 2023.

Zapechelnyuk, Andriy, "Optimal quality certification," *American Economic Review: Insights*, 2020, *2* (2), 161–176.

Appendix A Proofs

A.1 Derivation of the Consumer's Belief

The representative consumer rationally believes that the sellers are taking the linear strategy $F_i = \alpha_i\theta_i + \gamma_i$. That is, the consumer believes that the true quality θ_i and the rating R_i are interacting as follows:

$$\begin{aligned}\begin{bmatrix}\theta_i \\ R_i\end{bmatrix} &= \begin{bmatrix}\theta_i \\ \theta_i + F_i + \epsilon_i\end{bmatrix} = \begin{bmatrix}\theta_i \\ (1+\alpha_i)\,\theta_i + \gamma_i + \epsilon_i\end{bmatrix} \\ &= \begin{bmatrix}0 \\ \gamma_i\end{bmatrix} + \begin{bmatrix}1 & 0 \\ (1+\alpha_i) & 1\end{bmatrix}\begin{bmatrix}\theta_i \\ \epsilon_i\end{bmatrix} \\ &\sim N\left(\begin{bmatrix}\mu \\ (1+\alpha_i)\,\mu + \gamma_i\end{bmatrix}, \Omega\right),\end{aligned}$$

where

$$\begin{aligned}\Omega &= \begin{bmatrix}1 & 0 \\ (1+\alpha_i) & 1\end{bmatrix}\begin{bmatrix}\sigma_\theta^2 & 0 \\ 0 & \sigma_\epsilon^2\end{bmatrix}\begin{bmatrix}1 & (1+\alpha_i) \\ 0 & 1\end{bmatrix} \\ &= \begin{bmatrix}\sigma_\theta^2 & (1+\alpha_i)\,\sigma_\theta^2 \\ (1+\alpha_i)\,\sigma_\theta^2 & (1+\alpha_i)^2\,\sigma_\theta^2 + \sigma_\epsilon^2\end{bmatrix}.\end{aligned}$$

Thus, the required property is derived as the conditional expectation of multivariate normal distribution. Given this multivariate normal distribution, the reputation of firm i's product (Y_i) is written as

$$\underbrace{E\left[\theta_i | R_1,\ R_2\right]}_{=Y_i} = \mu + \underbrace{\frac{(1+\alpha_i)\,\sigma_\theta^2}{(1+\alpha_i)^2\,\sigma_\theta^2 + \sigma_\epsilon^2}}_{=\lambda_i}(R_i - \underbrace{\{(1+\alpha_i)\,\mu + \gamma_i\}}_{E[R_i]}),$$

where λ_i is a discounting parameter.

A.2 Proof of Proposition 3

Proof. On one hand, λ_i satisfying Equation (16) changes from 0 to $\sqrt{\phi_i/(2\beta_i^{-1}L_i^2)}$ as α_i changes from 0 to ∞. On the other hand, λ_i in Equation (17) changes from $1/(1+(\sigma_\epsilon/\sigma_\theta))$ to zero as α_i changes from 0 to ∞. Thus, those equations intersect each other at some point. The corresponding $E[F_i]$ is determined by Equation (18), and then, γ_i is determined by Equation (19). Thus, there exist α_i and γ_i which satisfy the equilibrium condition.

The uniqueness is obtained from the unique intersection of Equations (16) and (17) when $\sigma_\theta > \sigma_\epsilon$. With this parameter requirement, λ_i from eq. (17) is decreasing in $\alpha > 0$ while λ_i from eq.(16) is always increasing in α_i. □

Appendix B Equilibrium with a General Specification

In this section, we provide an equilibrium analysis for a general specification with n-firms, rational or naive belief by the consumer. Each of the 2-firm, n-symmetric-firm, rational belief, naive belief case is derived as a special case of the general specification.

B.1 Demand and Consumer Surplus

Given the observed ratings and prices, the consumer maximizes the following the interim expected utility by adjusting the consumption quantities, $\mathbf{q}$:

$$E_c\left[U|\mathbf{R},\mathbf{p}\right] = \left(\mathbf{Y}-\mathbf{p}\right)'\mathbf{q} - \frac{1}{2}\mathbf{q}'\mathbf{B}\mathbf{q} \tag{36}$$

where $\mathbf{Y}=(Y_1,\ ...,Y_n)'$ and $Y_i \equiv E_c\left[\theta_i|\mathbf{R},\mathbf{p}\right] = E_c\left[\theta_i|R_i\right]$. Then, by taking the first-order-condition with respect to $\mathbf{q}$:

$$\mathbf{p} = \mathbf{Y}-\mathbf{B}\mathbf{q} \tag{37}$$

$$\Leftrightarrow \mathbf{q} = \mathbf{B}^{-1}\left(\mathbf{Y}-\mathbf{p}\right) \tag{38}$$

Note that the expectation by the consumer at the point of the purchase decision, $E_c\left[\cdot|\mathbf{R},\mathbf{p}\right]$, can be potentially biased by the manipulative behavior depending on whether the consumer is rational or naive. To evaluate the consumer surplus, we insert the inverse demand function (37) into the ex-post utility function (36) and take the unbiased objective expectation of it.

$$\begin{aligned} CS = E\left[U\right] &= E\left[\left(\boldsymbol{\theta}-\left(\mathbf{Y}-\mathbf{B}\mathbf{q}\right)\right)'\mathbf{q} - \frac{1}{2}\mathbf{q}'\mathbf{B}\mathbf{q}\right] \\ &= \frac{1}{2}E\left[\mathbf{q}'\mathbf{B}\mathbf{q}\right] + E\left[\left(\boldsymbol{\theta}-\mathbf{Y}\right)'\mathbf{q}\right] \\ &= \frac{1}{2}tr\left(\mathbf{B}\Sigma_{\mathbf{q}}\right) + \frac{1}{2}E\left[\mathbf{q}\right]'\mathbf{B}E\left[\mathbf{q}\right] - E\left[\left(\mathbf{Y}-\boldsymbol{\theta}\right)'\mathbf{q}\right] \end{aligned}$$

where $\Sigma_{\mathbf{q}}$ is the variance-covariance matrix of $\mathbf{q}$. Note that the last term is the effect of the bias caused by sellers' manipulation. By the Law of Iterated Expectation, it is reduced to

$$E\left[\left(\mathbf{Y}-\boldsymbol{\theta}\right)'\mathbf{q}\right] = E\left[\left(\mathbf{Y}-E\left[\boldsymbol{\theta}|\mathbf{R},\mathbf{p}\right]\right)'\mathbf{q}\right].$$

This term disappears for the rational consumer because their conditional expectation matches with the objective conditional expectation, $Y \equiv E_c\left[\boldsymbol{\theta}|\mathbf{R},\mathbf{p}\right] = E\left[\boldsymbol{\theta}|\mathbf{R},\mathbf{p}\right]$. In contrast, if the consumer is naive, the reputation $\mathbf{Y}$ is inflated and greater than $E\left[\boldsymbol{\theta}|\mathbf{R},\mathbf{p}\right]$. Thus, the last term represents a negative impact of the manipulation.

B.2 Manipulation Strategy

Given the demand function in Equation (38), firms compete in prices or quantities. This leads to the equilibrium profit as a quadratic function of q_i^* and the equilibrium quantity as a linear combination of $(Y_i - c_i)$'s:

$$\pi_i = \beta_i^{-1} (q_i^*)^2 - \frac{\phi_i}{2} F_i^2 \tag{39}$$

$$q_i^* = L_i (Y_i - c_i) - M_i \sum_{j \neq i} (Y_j - c_j) \tag{40}$$

for some constants L_i and K_i. See the Appendix C for the derivation. Given the own quality θ_i, firm i maximizes a conditionally expected profit in the following

$$E[\pi_i | \theta_i, F_i] = \beta_i^{-1} \left\{ Var(q_i^* | \theta_i, F_i) + E[q_i^* | \theta_i, F_i]^2 \right\} - \frac{\phi_i}{2} F_i^2$$

Because Lemma 1 holds regardless of the number of firms or consumer's naivete, the first-order condition w.r.t. F_i is written as

$$0 = 2\beta_i^{-1} \frac{\partial E[q_i^* | \theta_i, F_i]}{\partial F_i} E[q_i^* | \theta_i, F_i] - \phi_i F_i$$

Here, recall that $q_i^* = L_i (Y_i - c_i) - M_i \sum_{j \neq i} (Y_j - c_j)$ and $E[Y_j | \theta_i, F_i] = E[Y_j]$, and note $\frac{\partial E\left[q_i^* | \theta_i, F_i\right]}{\partial F_i} = \frac{\partial E\left[q_i^* | \theta_i, F_i\right]}{\partial Y_i} \frac{\partial Y_i}{\partial F_i} = L_i \lambda_i$. Then, the marginal gain from the manipulation is decomposed into two parts regarding whether it depends on the hidden information of firm i (θ_i, F_i):

$$\begin{aligned} 0 &= 2\beta_i^{-1} L_i \lambda_i E\left[L_i (Y_i - c_i) - M_i \sum_{j \neq i} (Y_j - c_j) | \theta_i, F_i \right] - \phi_i F_i \\ &= 2\beta_i^{-1} L_i \lambda_i \left(E[q_i^*] + L_i \left(E[Y_i | \theta_i, F_i] - E[Y_i] \right) \right) - \phi_i F_i \qquad (41) \\ &= 2\beta_i^{-1} L_i \lambda_i \left(E[q_i^*] + L_i \lambda_i \left(\theta_i - E[\theta_i] + F_i - E[F_i] \right) \right) - \phi_i F_i \qquad (42) \end{aligned}$$

Thus, regardless of whether the consumer is rational ($E_c[F_i] = E[F_i]$) or naive ($E_c[F_i] = 0$), the firm i' optimal manipulation strategy F_i saisfies the above equation. By rearranging it,

$$\left(\phi - 2\beta_i^{-1} L_i^2 \lambda_i^2 \right) F_i = 2\beta_i^{-1} L_i^2 \lambda_i^2 \theta_i + 2\beta_i^{-1} L_i \lambda_i \left(E[q_i^*] + L_i \lambda_i \left(-E[\theta_i] - E[F_i] \right) \right)$$

On the right-hand side, the only term that varies with θ_i is the first term, which is linear in θ_i. Thus, for some γ_i, the manipulation strategy satisfies $F_i = \alpha_i\theta_i + \gamma_i$ where $\alpha_i = \frac{2\beta_i^{-1}L_i^2\lambda_i^2}{\phi_i - 2\beta_i^{-1}L_i^2\lambda_i^2}$. $E[F_i]$ is characterized by the expectation of the equation (42).

$$
\begin{aligned}
0 &= 2\beta_i^{-1}L_i\lambda_i\left(E\left[q_i^*\right] + L_i\lambda_i\left(E\left[\theta_i\right] - E\left[\theta_i\right] + E\left[F_i\right] - E\left[F_i\right]\right)\right) - \phi_i E\left[F_i\right] \\
&= 2\beta_i^{-1}L_i\lambda_i E\left[q_i^*\right] - \phi_i E\left[F_i\right]
\end{aligned}
$$

By stacking the above equation for all $i = 1, ..., n$,

$$
\begin{aligned}
\mathbf{0} &= 2\mathbf{diag}(\beta_i^{-1}L_i\lambda_i)E\left[\mathbf{q}\right] - \mathbf{diag}(\boldsymbol{\phi})E\left[\mathbf{F}\right] \\
&= 2\mathbf{diag}(\beta_i^{-1}L_i\lambda_i)\left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1}\left(E\left[\mathbf{Y}\right] - \mathbf{c}\right) - \mathbf{diag}(\boldsymbol{\phi})E\left[\mathbf{F}\right]
\end{aligned}
$$

In general, $E[\mathbf{Y}]$ can be a function of $E[\mathbf{F}]$. Explicitly,

$$
\begin{aligned}
Y_i &= E\left[\theta_i\right] + \frac{Cov_c\left(\theta_i,\, R_i\right)}{Var_c\left(R_i\right)}\left(R_i - E_c\left[R_i\right]\right) \\
&= E\left[\theta_i\right] + \lambda_i\left(\theta_i + F_i + \epsilon_i - E_c\left[\theta_i\right] - E_c\left[F_i\right]\right)
\end{aligned}
$$

and $E_c[\theta_i] = E[\theta_i]$ hold. Thus, the expectation of the reputation $E[Y_i]$ is written as

$$
E\left[Y_i\right] = E\left[\theta_i\right] + \lambda_i\left(E\left[F_i\right] - E_c\left[F_i\right]\right)
$$

Then, the above stacked FOC is written as a formula with $E[\mathbf{F}]$ as the only endogenous variable:

$$
\begin{aligned}
\mathbf{0} &= 2\mathbf{diag}(\beta_i^{-1}L_i\lambda_i)\left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1}\left(E\left[\boldsymbol{\theta}\right] + \mathbf{diag}(\lambda_i)\left(E\left[\mathbf{F}\right] - E_c\left[\mathbf{F}\right]\right) - \mathbf{c}\right) - \mathbf{diag}(\boldsymbol{\phi})E\left[\mathbf{F}\right] \\
&= 2\mathbf{diag}(\beta_i^{-1}L_i\lambda_i)\left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1}\left(E\left[\boldsymbol{\theta}\right] - \mathbf{c}\right) \\
&+ 2\mathbf{diag}(\beta_i^{-1}L_i\lambda_i)\left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1}\mathbf{diag}(\lambda_i)\left(E\left[\mathbf{F}\right] - E_c\left[\mathbf{F}\right]\right) - \mathbf{diag}(\boldsymbol{\phi})E\left[\mathbf{F}\right]
\end{aligned}
$$

If the consumer is rational (i.e., $E_c[\mathbf{F}] = E[\mathbf{F}]$), $E[\mathbf{F}]$ is characterized by

$$
\mathbf{diag}(\boldsymbol{\phi})E\left[\mathbf{F}\right] = 2\mathbf{diag}(\beta_i^{-1}L_i\lambda_i)\left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1}\left(E\left[\boldsymbol{\theta}\right] - \mathbf{c}\right) \tag{43}
$$

Note that the right-hand side is constant for each firm and $E[\mathbf{F}]$ on the left-hand side is multiplied by an diagonal matrix. Thus, the above equation is separated with each row as $\phi_i E[F_i] = \delta_i$ for a certain constant δ_i. Thus, a change of ϕ_j $(j \neq i)$ does not propagate to $E[F_i]$.

In contrast, if the consumer is naive, (i.e., $E_c[\mathbf{F}] = \mathbf{0}$), $E[\mathbf{F}]$ must satisfy

$$\Big(\mathbf{diag}(\boldsymbol{\phi}) - 2\mathbf{diag}(\beta_i^{-1}L_i\lambda_i)\left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1}\mathbf{diag}(\lambda_i)\Big)E[\mathbf{F}] \tag{44}$$

$$= 2\mathbf{diag}(\beta_i^{-1}L_i\lambda_i)\left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1}(E[\boldsymbol{\theta}] - \mathbf{c}) \tag{45}$$

Now, on the left-hand side, $E[\mathbf{F}]$ is multiplied by a non-diagonal matrix. Thus, a change of ϕ_j $(j \neq i)$ propagates to $E[F_i]$ via a strategic concern in the manipulation stage. More specifically, if the products are substitute $(s > 0)$ and only two firms or n symmetric firms are competing in the market, $E[\mathbf{F}]$ is multiplied by a matrix with only positive elements. The signs of the off-diagonal elements are verified by the espression in Subsection C, and the sign of the diagonal elements are supposed to be positive for the second-order condition of each firm's profit maximization in F_i. Thus, $E[F_i]$ is expressed as a linear decreasing function of $E[F_j]$ $(j \neq i)$, and then, the manipulation stage has a strategic substitute property in $E[F_i]$'s. Thus, a high ϕ_j implies a low $E[F_j]$, which then implies a high $E[F_i]$.

For completeness of the equilibrium characterization, γ_i is characterized by

$$\gamma_i = E[F_i] - \alpha_i E[\theta_i].$$

Appendix C Derivation of L_i and M_i

This subsection derives L_i and M_i from the pricing stage. First, we discuss the general case and introduce 2-firm case and n-symmetric-firm case as special cases. L_i and M_i in a quantity competition are also derived by a slight modification of the derivation.

C.1 Bertrand Equilibrium

Once firms observe all ratings, and given the other firms' prices, each firm expects that the consumer forms a reputation vector $\mathbf{Y}$ and behaves following the demand function (38). Then, try to maximize $(p_i - c_i)\, q_i - \frac{\phi_i}{2} F_i$ by adjusting its price. The first-order condition w.r.t. p_i is

$$0 = (p_i - c_i)\frac{\partial q_i}{\partial p_i} + q_i = -\beta_i (p_i - c_i) + q_i \tag{46}$$

where β_i is the i-i-element of $\mathbf{B}^{-1}$. By stacking first-order conditions for all $i = 1, ..., n$,

$$\begin{aligned} &\mathbf{0} = -\mathbf{diag}(\beta)\,(\mathbf{p} - \mathbf{c}) + \mathbf{q} \\ \Leftrightarrow\ &\mathbf{0} = \mathbf{Y} - \mathbf{Bq} - \mathbf{c} - \mathbf{diag}(\beta)^{-1}\mathbf{q} \\ \Leftrightarrow\ &\mathbf{q} = \left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1}(\mathbf{Y} - \mathbf{c}) \equiv \mathbf{q}^* \end{aligned}$$

Furthermore, (46) implies

$$\begin{aligned} &\mathbf{p}^* - \mathbf{c} = \mathbf{diag}(\beta)^{-1}\mathbf{q}^* \\ \Leftrightarrow\ &p_i^* - c_i = \beta_i^{-1} q_i^* \text{ for all } i \end{aligned}$$

Thus, the firm's profit from the price competition stage given the realized rating is $\pi_i = \beta_i^{-1}(q_i^*)^2 - \frac{\phi_i}{2}F_i^2$.

Two Firm Example Because q_i^* is a linear combination of $(Y_i - c_i)$ and $(Y_j - c_j)$, the profit function is rewritten as $\pi_i = J_i\left((Y_i - c_i) - K_i(Y_j - c_j)\right)^2 - \frac{\phi_i}{2}F_i^2$ for some constans J_i and K_i. More explicitly, let

$$\mathbf{B} = \begin{bmatrix} b_1 & s \\ s & b_2 \end{bmatrix}.$$

Then,

$$\mathbf{B}^{-1} = \frac{1}{b_1 b_2 - s^2}\begin{bmatrix} b_2 & -s \\ -s & b_1 \end{bmatrix}, \mathbf{diag}(\beta) = \frac{1}{b_1 b_2 - s^2}\begin{bmatrix} b_2 & 0 \\ 0 & b_1 \end{bmatrix}$$

and

$$\left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right) = \begin{bmatrix} b_1 + \beta_1^{-1} & s \\ s & b_2 + \beta_2^{-1} \end{bmatrix}$$

$$\left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1} = \frac{1}{\left(b_1 + \beta_1^{-1}\right)\left(b_2 + \beta_2^{-1}\right) - s^2} \begin{bmatrix} b_2 + \beta_2^{-1} & -s \\ -s & b_1 + \beta_1^{-1} \end{bmatrix}$$

Thus,

$$q_i^* = \frac{b_j + \beta_j^{-1}}{\left(b_1 + \beta_1^{-1}\right)\left(b_2 + \beta_2^{-1}\right) - s^2}(Y_i - c_i) - \frac{s}{\left(b_1 + \beta_1^{-1}\right)\left(b_2 + \beta_2^{-1}\right) - s^2}(Y_j - c_j)$$

Note that

$$b_i + \beta_i^{-1} = b_i + \frac{b_1 b_2 - s^2}{b_j} = \frac{2b_1 b_2 - s^2}{b_j}$$

Thus,

$$\begin{aligned} q_i^* &= \frac{1}{\frac{2b_1b_2 - s^2}{b_2}\frac{2b_1b_2 - s^2}{b_1} - s^2}\left(\frac{2b_1b_2 - s^2}{b_i}(Y_i - c_i) - s(Y_j - c_j)\right) \\ &= \frac{b_1 b_2}{(4b_1b_2 - s^2)(b_1b_2 - s^2)}\left(\frac{2b_1b_2 - s^2}{b_i}(Y_i - c_i) - s(Y_j - c_j)\right) \\ &= L_i(Y_i - c_i) - M_i(Y_j - c_j) \end{aligned}$$

where $L_i = \frac{b_j\left(2b_1b_2 - s^2\right)}{(4b_1b_2 - s^2)(b_1b_2 - s^2)}$ and $M_i = \frac{b_1b_2s}{(4b_1b_2 - s^2)(b_1b_2 - s^2)}$.

***n*-firm Example with Symmetric** For n-firm case with symmetric substitution pattern, by employing an n-by-n identity matrix **I** and an n-by-n matrix **J** with 1's in all elements (as in Chone and Linemmer (2020)), **B** is written as

$$\mathbf{B} = (b - s)\,\mathbf{I} + s\mathbf{J}$$

and its iverse is characterized as

$$\mathbf{B}^{-1} = \frac{1}{(b - s)}\left(\mathbf{I} - \frac{s}{b + (n-1)\,s}\mathbf{J}\right)$$

which is easily verified by $\mathbf{BB}^{-1} = \mathbf{I}$. Thus, for any $i = 1, \ldots, n$,

$$\beta_i = \left[\mathbf{B}^{-1}\right]_{ii} = \frac{1}{(b-s)} \left(\frac{b + (n-2)\, s}{b + (n-1)\, s} \right)$$
$$\Leftrightarrow \beta_i^{-1} = \frac{(b-s)\,(b + (n-1)\, s)}{b + (n-2)\, s}.$$

Then,

$$\mathbf{B} + \mathbf{diag}(\beta)^{-1} = (b-s)\,\mathbf{I} + s\mathbf{J} + \beta_i^{-1}\mathbf{I} = \left(b - s + \beta_i^{-1}\right)\mathbf{I} + s\mathbf{J}$$
$$\Leftrightarrow \left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1} = \frac{1}{\beta_i^{-1} + b - s} \left(\mathbf{I} - \frac{s}{\beta_i^{-1} + b + (n-1)\, s} \mathbf{J} \right)$$

Therefore

$$\begin{aligned} q_i^* &= \frac{1}{\beta_i^{-1} + b - s} \left(Y_i - c_i - \frac{s \sum_j (Y_j - c_j)}{\beta_i^{-1} + b + (n-1)\, s} \right) \\ &= \frac{1}{\left(\beta_i^{-1} + b - s\right)\left(\beta_i^{-1} + b + (n-1)\, s\right)} \left(\left(\beta_i^{-1} + b + (n-2)\, s\right)(Y_i - c_i) - s \left(\sum_{j \neq i} (Y_j - c_j) \right) \right) \\ &= L_i\,(Y_i - c_i) - M_i \left(\sum_{j \neq i} (Y_j - c_j) \right) \end{aligned}$$

where $L_i = \frac{\left(\beta_i^{-1} + b + (n-2)s\right)}{\left(\beta_i^{-1} + b - s\right)\left(\beta_i^{-1} + b + (n-1)s\right)}$ and $M_i = \frac{s}{\left(\beta_i^{-1} + b - s\right)\left(\beta_i^{-1} + b + (n-1)s\right)}$.

C.2 Cournot Equilibrium

If firm i believes that the consumer's behavior is characterized by the inverse demand function (37), and if it adjust the own quantity given the others' quantity, the first-order condition with respect to q_i is

$$0 = \frac{\partial p_i}{\partial q_i} q_i + (p_i - c_i) = -b_i q_i + (p_i - c_i) \tag{47}$$

where b_i is the i-i-element of B. By stacking first-order conditions for all $i = 1, \ldots, n$,

$$\mathbf{0} = (\mathbf{p} - \mathbf{c}) - \mathbf{diag}(\mathbf{b})\mathbf{q}$$
$$\Leftrightarrow \mathbf{0} = \mathbf{Y} - \mathbf{Bq} - \mathbf{c} - \mathbf{diag}(\mathbf{b})\mathbf{q}$$
$$\Leftrightarrow \mathbf{q} = \left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1}(\mathbf{Y} - \mathbf{c}) \equiv \mathbf{q}^C$$

Thus, we can obtain L_i and M_i for Cournot competition by replacing β_i^{-1} with b_i. More explicitly, $L_i = \frac{2b_j}{4b_1 b_2 - s^2}$ and $M_i = \frac{s}{4b_1 b_2 - s^2}$ for 2-firms Cournot, and $L_i = \frac{(2b + (n-2)s)}{(2b - s)(2b + (n-1)s)}$ and

$M_i = \frac{s}{(2b-s)(2b+(n-1)s)}$ for n-firm Cournot competition with symmetric substitution.

Appendix D Limit Analysis

Proposition 12 (Limits of the Equilibrium)**.** *The equilibrium strategy has the following properties:*

1. *As $s \to 0$, the second stage is reduced to the monopoly for each product (i.e., Dellarocas (2006)).*
2. *As $s \to b_1 = b_2 = b$, $\alpha_i \to \infty$ and $\lambda \to 0$ in price competition, and $\alpha_i, \lambda_i \to \bar{\alpha}, \bar{\lambda}$ for some finite $\bar{\alpha}, \bar{\lambda}$ in quantity competition.*
3. *As $\phi_i \to \infty$, $\alpha_i, \gamma_i \to 0$. (Thus, $E[F_i] \to 0$.)*
4. *As $\sigma_\theta/\sigma_\epsilon \to \infty$, α_i and $\lambda \to \hat{\alpha}, \hat{\lambda}$ for some finite $\hat{\alpha}, \hat{\lambda}$ and $Cor(\theta_i, Y_i) \to 1$*
5. *As $\sigma_\theta/\sigma_\epsilon \to 1$, α_i and $\lambda \to \tilde{\alpha}, \tilde{\lambda}$ for some finite $\tilde{\alpha}, \tilde{\lambda}$*

Appendix E Partially Naive Consumer

E.1 Definition of the Partially Naive Consumer

If the consumer is partially rational with degree η, the product i's reputation can be defined as follows:

$$
\begin{aligned}
E_c[\theta_i|R_i] = Y_i^\eta &\equiv \eta Y_i^R + (1-\eta)\, Y_i^N \\
&= \eta \left\{ E[\theta_i] + \lambda_i^R \left(R_i - \{E[\theta_i] + E[F_i]\}\right)\right\} + (1-\eta)\left\{ E[\theta_i] + \lambda_i^N \left(R_i - E[\theta_i]\right)\right\} \\
&= E[\theta_i] + \lambda_i^\eta \left(R_i - E[\theta_i]\right) - \eta \lambda_i^R E\left[F_i\right]
\end{aligned}
$$

where Y_i^R and Y_i^N are the reputation formed by a rational and naive consumer, respectively, and $\lambda_i^\eta \equiv \eta\lambda_i^R + (1-\eta)\,\lambda_i^N$. Such a convex combination of beliefs can be interpreted that either (i) the representative consumer (each consumer in a mass) is partially rational, or (ii) there are a mass η of rational consumers and a mass $(1-\eta)$ of naive consumers.

Given the above definition of the partially naive consumer's belief, we can still characterize the equilibrium. The rationality parameter η smoothly connects the results of two extremes of rational and naive.

E.2 Interpretation of the Partially Naive Consumer

If the market is filled with a mass η of rational consumers and a mass $(1-\eta)$ of naive consumers, the market for product i is defined as $q_i = \eta q_i^R + (1-\eta)\, q_i^N$ where q_i^R is the demand from a rational consumer and q_i^N is the demand from a naive consumer, each of which is defined so far. Then, the market demand for product i is

$$
\begin{aligned}
q_i &= \eta q_i^R + (1-\eta)\, q_i^N \\
&= \eta \left\{ \frac{b_j Y_i^R - s Y_j^R}{b_i b_j - s^2} - \frac{b_j}{b_i b_j - s^2} p_i + \frac{s}{b_i b_j - s^2} p_j \right\} \\
&+ (1-\eta) \left\{ \frac{b_j Y_i^N - s Y_j^N}{b_i b_j - s^2} - \frac{b_j}{b_i b_j - s^2} p_i + \frac{s}{b_i b_j - s^2} p_j \right\} \\
&= \frac{b_j Y_i^\eta - s Y_j^\eta}{b_i b_j - s^2} - \frac{b_j}{b_i b_j - s^2} p_i + \frac{s}{b_i b_j - s^2} p_j
\end{aligned}
$$

where $Y_i^\eta = \eta Y_i^R + (1-\eta)\, Y_i^N$. Thus, the demand function is equivalent to the one from the partially rational representative consumer.

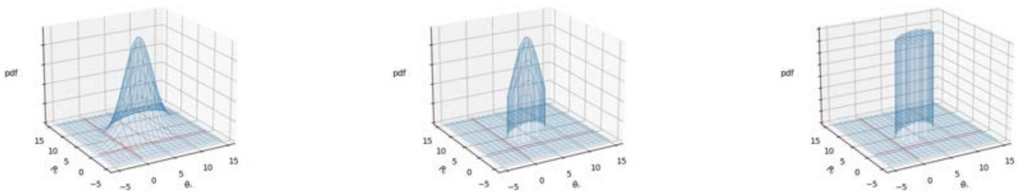

Figure 7: Examples of Elliptical Distribution

Appendix F Model with a Joint Elliptical Distribution

Note that we actively use the normality assumption when we calculate the rating as a linear combination of quality and noise, and we calculate the conditional expectation of quality given the rating's realization. Those properties of the jointly normal distribution are robust in the general elliptical distribution in a sense explained below. For the normative analysis in Subsection 4.1, the surplus functions, which is characterized by second moments, are just multiplied by a scalar. Therefore, the ordinal property of the surplus function is robust with the general elliptical distribution as well. Thus, the results in the main part is robust with a non-negative elliptical distribution whose tail is truncated.

Suppose that $x_i = \begin{bmatrix} \theta_i \\ \epsilon_i \end{bmatrix}$ follows elliptical distribution with a location parameter $\bar{\mu} = \begin{bmatrix} \mu_i \\ 0 \end{bmatrix}$, scale parameter $\Sigma = \begin{bmatrix} \sigma_\theta^2 & 0 \\ 0 & \sigma_\epsilon^2 \end{bmatrix}$, and a density function $g(\cdot)$ (Cambanis et al., 1981; Gómez et al., 2003). That is, the probability density function of $\begin{bmatrix} \theta_i \\ \epsilon_i \end{bmatrix}$ is characterized as $f_i(x_i) = g_i(x_i'\Sigma x_i)$. More precisely, we suppose that x_i and x_j are jointly follow an elliptial distribution with Σ's as block diagonal elements and 0's off-diagonal elements of scaling parameters. Therefore, (i) x_i and x_j are not correlated, (ii) an affine transformation of $[x_i, x_j]'$ follows an elliptical distribution, whose location and scaling parameters are characterized as in a joint normal distribution, but (iii) x_i and x_j are not necessarily independent each other (e.g., If the support of (θ_1, θ_2) is an area enclosed by a circle with a finite radius, the support of $(\theta_1|\theta_2)$ depends on the value of θ_2).

Let a rating on Seller i as a noisy and potentially biased signal of the underlying quality, $R_i = \theta_i + F_i + \epsilon_i$. Suppose that Seller i uses a linear strategy in fake reviews, that is,

$$F_i = \alpha_i \theta_i + \gamma_i$$

Then, a rating on Seller i's product is written as

$$R_i = \theta_i + \alpha_i \theta_i + \gamma_i + \epsilon_i$$

Now, $\begin{bmatrix} \theta_i \\ R_i \end{bmatrix}$ is written as a linear transformation of $\begin{bmatrix} \theta_i \\ \epsilon_i \end{bmatrix}$:

$$\begin{bmatrix} \theta_i \\ R_i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1+\alpha_i & 1 \end{bmatrix} \begin{bmatrix} \theta_i \\ \epsilon_i \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma_i \end{bmatrix}$$

Then, by Theorem 5 of Gomez et al (2003), $\begin{bmatrix} \theta_i \\ R_i \end{bmatrix}$ follows an elliptical distribution with a location parameter, $\begin{bmatrix} 1 & 0 \\ 1+\alpha_i & 1 \end{bmatrix} \begin{bmatrix} \mu_i \\ \epsilon_i \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma_i \end{bmatrix} = \begin{bmatrix} \mu_i \\ (1+\alpha_i)\mu_i + \gamma_i \end{bmatrix}$, scaling parameter

$$\begin{bmatrix} 1 & 0 \\ (1+\alpha_i) & 1 \end{bmatrix} \begin{bmatrix} \sigma_\theta^2 & 0 \\ 0 & \sigma_\epsilon^2 \end{bmatrix} \begin{bmatrix} 1 & (1+\alpha_i) \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \sigma_\theta^2 & \sigma_\theta^2 (1+\alpha_i) \\ \sigma_\theta^2 (1+\alpha_i) & \sigma_\theta^2 (1+\alpha_i)^2 + \sigma_\epsilon^2 \end{bmatrix}$$

with some density function $\tilde{g}_i(\cdot)$.

Then, by Theorem 8 of Gómez et al. (2003), $(\theta_i | R_i)$ follows an elliptical distribution with a location parameter

$$\mu_i + \frac{\sigma_\theta^2 (1+\alpha_i)}{\sigma_\theta^2 (1+\alpha_i)^2 + \sigma_\epsilon^2} \left(R_i - ((1+\alpha_i)\mu_i + \gamma_i)\right)$$

scaling parameter $\sigma_\theta^2 - \frac{\sigma_\theta^4 (1+\alpha_i)^2}{\sigma_\theta^2 (1+\alpha_i)^2 + \sigma_\epsilon^2}$ with some density function $\hat{g}(\cdot)$.